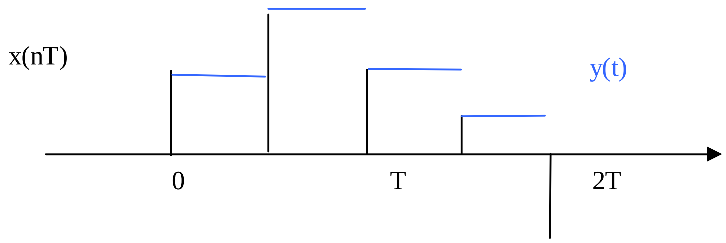
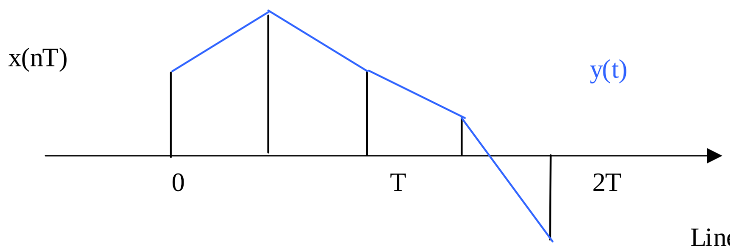
HW10

1. Given the following discrete-time input x(nT), sketch the outputs reconstructed continuous signal y(t) using Sample-And-Hold and Linear Interpolation.

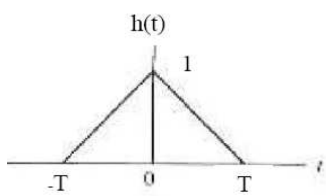



Sample-and-Hold

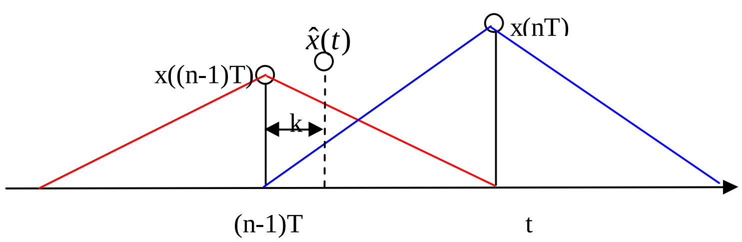



Linear interpolation

2. Show that the impulse response for the low-pass filter used in linear interpolation is as follows, where T is the sampling period:




Let's say t = (n-1)T + k with 0<k<T




Using graphical convolution, we have

Contribution from $x((n-1)T) = x((n-1)T)\dfrac{T-k}{T}$ (Red part)

Contribution from $x(nT) = x(nT)\dfrac{k}{T}$ (Blue part)

and there are no contribution from any other samples. Thus

$$x((n-1)T+k) = x((n-1)T)\frac{T-k}{T} + x(nT)\frac{k}{T}$$

which is clearly a linear combination of the closest two samples, weighted based on how close they are to the point of interpolation.

3. Show that the discrete-time complex exponentials $u[n] = e^{j\omega nT}$ is an eigen-signal for any discrete-time LTI system L, i.e. the following statement is true:

$$L\left[e^{j\omega nT}\right] = H(\omega)e^{j\omega nT}$$

Show that H(ω) is in fact the Discrete-Time Fourier Transform $H_d(\omega)$. Use this fact to show the DTFT of convolution x[n]*y[n] is the product of their respective DTFT $X_d(\omega)Y_d(\omega)$.

Assume L[.] is a discrete-time LTI system with $u[n] = e^{j\omega nT}$ as input. Since the zero-state output of L[.] can be computed by convolving the input with the impulse response h[n], we must have

$$y[n] = L\left[e^{j\omega nT}\right] = \sum_{m=-\infty}^{\infty} h[m]e^{j\omega(n-m)T} = e^{j\omega nT}\sum_{m=-\infty}^{\infty} h[m]e^{-j\omega mT} = e^{j\omega nT}H_d(\omega)$$

Except for the use of summation instead of integration, the proof here is exactly the same as that in the continuous-time case.

To prove the second part, let's assume that x[n] is the impulse response and y[n] is the input. Using the inverse DTFT formula on x[n], we can write x[n] as a linear combination (integration) of weighted complex exponentials:

$$x[n] = \frac{T}{2\pi}\int_0^{2\pi/T} X_d(\omega)e^{j\omega nT}d\omega$$

The first part shows that the output to each complex exponential is just $Y_d(\omega)e^{j\omega nT}$, so we have

$$x[n]*y[n] = \frac{T}{2\pi}\int_0^{2\pi/T} X_d(\omega)Y_d(\omega)e^{j\omega nT}d\omega = DTFT^{-1}\left[X_d(\omega)Y_d(\omega)\right]$$

4. During lecture, we discussed the Discrete Fourier transform pair:

$$X_k = \sum_{n=0}^{N-1} x(nT)e^{-j\frac{2\pi k}{N}n}, \quad k = 0,1,..,N-1$$

$$x(nT) = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi k}{N}n}, \quad n = 0,1,\ldots,N-1$$

You are given the discrete-time signal x[n] = [1 2 1], starting at n=0. Compute the four-point DFT (N=4) and its inverse. Do you get back the original signal? Now repeat the process for a two-point DFT (N=2).

The four point DFT is as follows:

$$X_0 = \sum_{n=0}^{2} x(nT)e^{-j\frac{2\pi 0}{4}n} = 1\cdot 1 + 2\cdot 1 + 3\cdot 1 = 6$$

$$X_1 = \sum_{n=0}^{2} x(nT)e^{-j\frac{2\pi 1}{4}n} = 1\cdot e^{-j\frac{\pi}{2}0} + 2\cdot e^{-j\frac{\pi}{2}1} + 3\cdot e^{-j\frac{\pi}{2}2} = 1\cdot 1 + 2\cdot(-j) + 3\cdot(-1) = -2-2j$$

$$X_2 = \sum_{n=0}^{2} x(nT)e^{-j\frac{2\pi 2}{4}n} = 1\cdot e^{-j\pi 0} + 2\cdot e^{-j\pi 1} + 3\cdot e^{-j\pi 2} = 1\cdot 1 + 2\cdot(-1) + 3\cdot 1 = 2$$

$$X_3 = \sum_{n=0}^{2} x(nT)e^{-j\frac{2\pi 3}{4}n} = 1\cdot e^{-j\frac{3\pi}{2}0} + 2\cdot e^{-j\frac{3\pi}{2}1} + 3\cdot e^{-j\frac{3\pi}{2}2} = 1\cdot 1 + 2\cdot j + 3\cdot(-1) = -2+2j$$

The four-point inverse DFT is as follows:

$$x[0] = \frac{1}{4}\sum_{k=0}^{3} X_k e^{j\frac{2\pi 0}{4}k} = \frac{1}{4}\left[6\cdot 1 + (-2-2j)\cdot 1 + 2\cdot 1 + (-2+2j)\cdot 1\right] = 1$$

$$x[1] = \frac{1}{4}\sum_{k=0}^{3} X_k e^{j\frac{2\pi 1}{4}k} = \frac{1}{4}\left[6\cdot e^{j\frac{\pi}{2}0} + (-2-2j)\cdot e^{j\frac{\pi}{2}1} + 2\cdot e^{j\frac{\pi}{2}2} + (-2+2j)\cdot e^{j\frac{\pi}{2}3}\right] = 2$$

$$x[2] = \frac{1}{4}\sum_{k=0}^{3} X_k e^{j\frac{2\pi 2}{4}k} = \frac{1}{4}\left[6\cdot e^{j\pi 0} + (-2-2j)\cdot e^{j\pi 1} + 2\cdot e^{j\pi 2} + (-2+2j)\cdot e^{j\pi 3}\right] = 3$$

$$x[3] = \frac{1}{4}\sum_{k=0}^{3} X_k e^{j\frac{2\pi 3}{4}k} = \frac{1}{4}\left[6\cdot e^{j\frac{3\pi}{2}0} + (-2-2j)\cdot e^{j\frac{3\pi}{2}1} + 2\cdot e^{j\frac{3\pi}{2}2} + (-2+2j)\cdot e^{j\frac{3\pi}{2}3}\right] = 0$$

We got back our original signal -- notice that the last sample x[3] becomes zero.

Given a sequence with three samples, it is odd to take only a 2-point DFT. Let's do it anyway by having three terms in the DFT, and compute three output samples using the IDFT:

$$X_0 = \sum_{n=0}^{2} x(nT)e^{-j\frac{2\pi 0}{2}n} = 1\cdot 1 + 2\cdot 1 + 3\cdot 1 = 6$$

$$X_1 = \sum_{n=0}^{2} x(nT)e^{-j\frac{2\pi 1}{2}n} = 1\cdot e^{-j\pi 0} + 2\cdot e^{-j\pi 1} + 3\cdot e^{-j\pi 2} = 1\cdot 1 + 2\cdot(-1) + 3\cdot 1 = -2$$

These two frequency samples are the first and third frequency components of the 4-point DFT – frequency sampling in action!

$$x[0] = \frac{1}{2}\sum_{k=0}^{1} X_k e^{j\frac{2\pi 0}{2}k} = \frac{1}{2}\left[6\cdot 1 + (-2)\cdot 1\right] = 2$$

$$x[1] = \frac{1}{2}\sum_{k=0}^{1} X_k e^{j\frac{2\pi 1}{2}k} = \frac{1}{2}\left[6\cdot e^{j\pi 0} + (-2)\cdot e^{j\pi 1}\right] = 4$$

$$x[2] = \frac{1}{2}\sum_{k=0}^{1} X_k e^{j\frac{2\pi 2}{2}k} = \frac{1}{2}\left[6\cdot e^{j2\pi 0} + (-2)\cdot e^{j2\pi 1}\right] = 2$$

We fail to get back the original sequence. Note that x[2] is the same as x[0]. You can try compute x[3], which will be the same as x[1]. The inverse 2-point DFT produces a periodic DT sequence of period 2.

Problem 6.2

Solution:

(a)

$$2s^2Y(s)+4sY(s)+10Y(s)=s^2U(s)-sU(s)-2U(s)$$

$$H(s)=\frac{Y(s)}{U(s)}=\frac{s^2-s-2}{2s^2+4s+10}$$

(b)

$$2s^3Y(s)+4s^2Y(s)+10sY(s)=s^2U(s)+3sU(s)+2U(s)$$

$$H(s)=\frac{Y(s)}{U(s)}=\frac{s^2+3s+2}{2s^3+4s^2+10s}$$

(c)

$$s^5Y(s)+2Y(s)=s^3U(s)+U(s)$$

$$H(s)=\frac{Y(s)}{U(s)}=\frac{s^3+1}{s^5+2}$$

Problem 6.3

Solution:

$$s^4V(s)+3s^3V(s)+10V(s)=2s^2R(s)+5sR(s)+3R(s)$$

$$v^{(4)}(t)+3v^{(3)}(t)+10v(t)=2\ddot{r}(t)+5\dot{r}(t)+3r(t)$$

Problem 6.4

Solution:

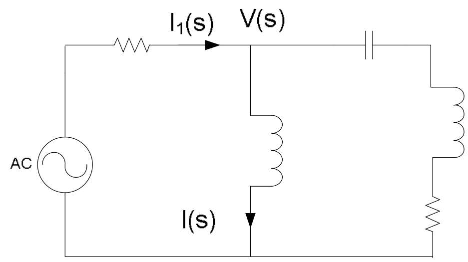
$$U(s)=\frac{2}{s^2+4}$$

$$Y(s)=\frac{4}{s+1}+\frac{2\times 2}{s^2+4}-\frac{4s}{s^2+4}=\frac{20}{(s+1)(s^2+4)}$$

$$H(s)=\frac{Y(s)}{U(s)}=\frac{10}{s+1}$$

Problem 6.6

Solution:




The impedance of the parallel connection of $2s$ and $\left(\frac{1}{3s}+3s+2\right)$ is

$$Z_{12}(s)=\frac{2s\left(\frac{1}{3s}+3s+2\right)}{2s+\left(\frac{1}{3s}+3s+2\right)}=\frac{2s\left(9s^2+6s+1\right)}{15s^2+6s+1}$$

$$I_1(s)=\frac{U(s)}{4+Z_{12}(s)}=\frac{15s^2+6s+1}{18s^3+72s^2+26s+4}U(s)$$

$$V(s)=I_1(s)Z_{12}(s)=\frac{2s\left(9s^2+6s+1\right)}{18s^3+72s^2+26s+4}U(s)$$

$$Y(s)=\frac{3s+2}{\frac{1}{3s}+3s+2}V(s)=\frac{3s(3s+2)}{9s^2+6s+1}V(s)$$

$$\frac{Y(s)}{U(s)}=\frac{6s^2(3s+2)}{18s^3+72s^2+26s+4}$$

$$I(s)=\frac{V(s)}{2s}$$

$$\frac{I(s)}{U(s)}=\frac{9s^2+6s+1}{18s^3+72s^2+26s+4}$$

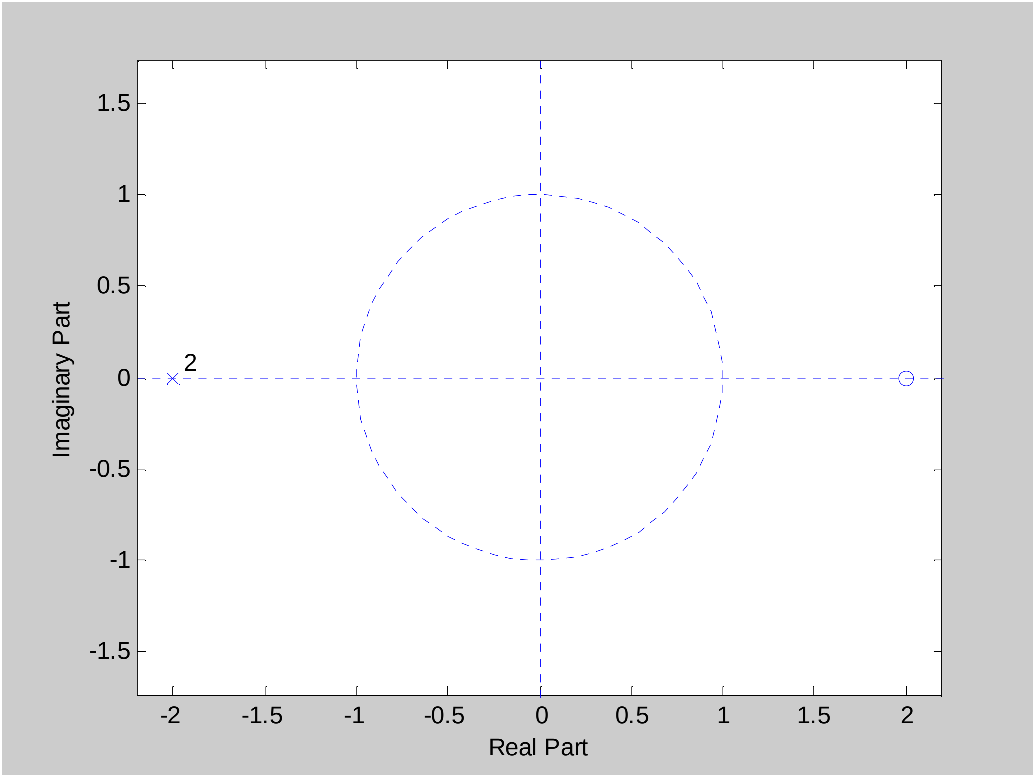
Problem 6.8

Solution:

(a)

$$H_1(s)=\frac{1.5(s-2)}{(s+2)^2}$$

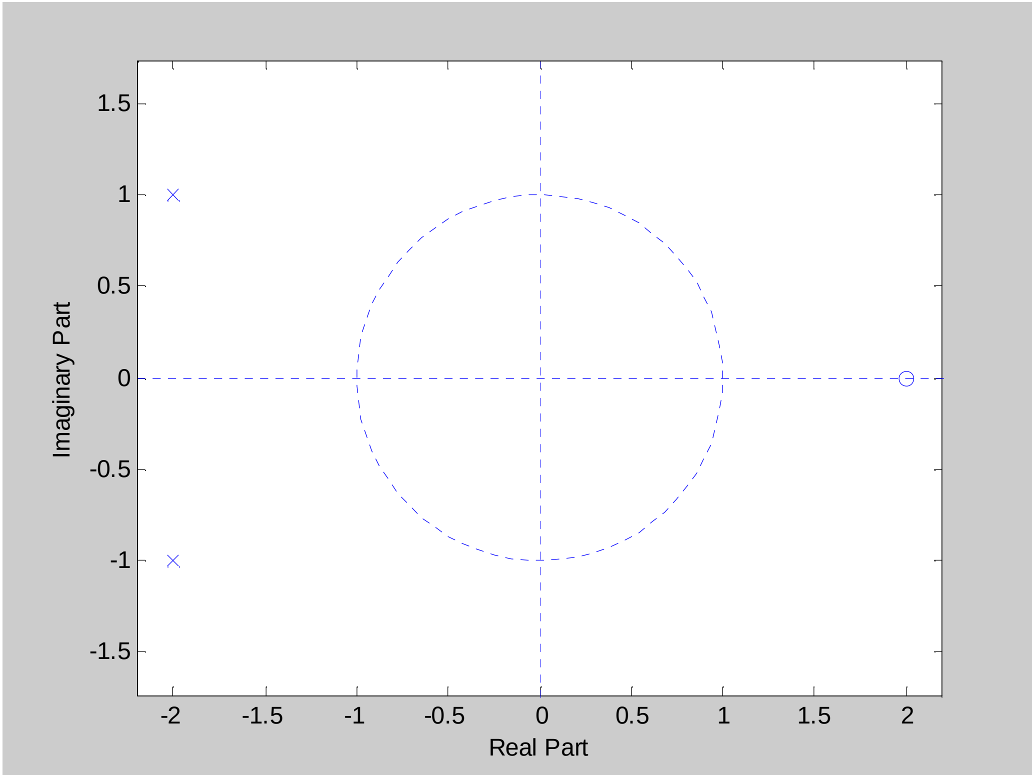
It has a simple zero at 2 and a repeated pole at -2 with multiplicity 2.




(b)

$$H_2(s) = \frac{0.5(s-2)}{(s+2+j1)(s+2-j1)}$$

It has a simple zero at 2 and simple poles at -2+j1 and -2-j1. Note that -1 is neither a pole nor a zero.




Problem 6.10

Solution:

$$u(t)=\delta(t),\ \ U(s)=1$$

$$Y(s)=H(s)U(s)=\frac{s^2+3}{(s+1)(s+2)(s-1)}$$

$$=\frac{-2}{s+1}+\frac{7/3}{s+2}+\frac{2/3}{s-1}$$

Impulse response = $-2e^{-t}+\frac{7}{3}e^{-2t}+\frac{2}{3}e^{t}$, for $t\geq 0$

$$u(t)=q(t),\ \ U(s)=\frac{1}{s}$$

$$Y(s)=H(s)U(s)=\frac{s^2+3}{(s+1)(s+2)(s-1)}\cdot\frac{1}{s}$$

$$=\frac{2}{s+1}-\frac{7/6}{s+2}+\frac{2/3}{s-1}-\frac{3/2}{s}$$

Step response = $2e^{-t}-\frac{7}{6}e^{-2t}+\frac{2}{3}e^{t}-\frac{3}{2}$, for $t\geq 0$

Problem 6.14

Solution:

$$Y(s) = H(s)\cdot\frac{1}{s} = \frac{N(s)}{s(s+2)^4(s+0.1)(s^2+2s+10)}$$

$$s^2+2s+10 = (s+1)^2+3^2$$

$$Y(s) = k_0 + \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{(s+2)^2} + \frac{\overline{k_4}}{(s+2)^3} + \frac{\overline{k_5}}{(s+2)^4} + \frac{k_6}{s+0.1}$$

$$+\overline{k_7}\frac{3}{(s+1)^2+3^2} + \overline{k_8}\frac{s+1}{(s+1)^2+3^2}$$

$k_0 = 0$ because $N(s)$ has degree 7 or less and $H(s)\cdot\frac{1}{s}$ is strictly proper.

$$k_1 = H(0) = \frac{N(0)}{2^4\times 0.1\times 10} = \frac{320}{16} = 20$$

$$\therefore y(t) = 20 + k_2e^{-2t} + k_3te^{-2t} + k_4t^2e^{-2t} + k_5t^3e^{-2t} + k_6e^{-0.1t} + k_7e^{-t}\sin(3t+k_8),\ \ t\ge 0$$