University of Kentucky
Department of Electrical and Computer Engineering

EE421G: Signals and Systems I – Fall 2007

Problem Set 11

Issued: November 25, 2007 Due: November 30, 2007 (In class)

Reading Assignments:
Read Chapter 6.7-6.11 of Chen

Paper and Pencil Assignments:

1. Problem 6.15

Solution:

$u(t) = \sin 10t,\ U(s) = \dfrac{10}{s^2+100}$

$Y(s) = \dfrac{10}{(s+1)(s^2+100)^2}$

$y(t) = k_1 e^{-t} + k_2 \sin(10t + k_3) + k_4 t \sin(10t + k_5)$

$y(t) \to \infty$ as $t \to \infty$. Theorem 6.4 does not hold.

2. Problem 6.17

Solution:

$U(s) = \dfrac{s-1.2}{s(s+1)}$

$u(t) = k_1 + k_2 e^{-t}, t \geq 0$ It is bounded.

$Y(s) = H(s)U(s) = \dfrac{1}{(s+2)(s-1.2)} \cdot \dfrac{(s-1.2)}{s(s+1)} = \dfrac{1}{(s+2)s(s+1)}$

$y(t) = \bar{k}_1 + \bar{k}_2 e^{-2t} + \bar{k}_3 e^{-t}, t \geq 0$

It does not contain $e^{1.2t}$, which is the response of the unstable pole of $H(s)$. Thus $y(t)$ is bounded. If $U(s)$ does not contain the zero $(s-1.2)$, then $y(t)$ will contain $e^{1.2t}$ and will grow unbounded.

3. Problem 6.20

Solution:

(a) Not stable because of missing term s^4.

(b) Not stable because of $-3s^3$.

(c) Not stable because a negative number appears.

	s^5	1 3 1	
k_1=1/4	s^4	4 2 1	[0 2.5 0.75]
k_2=1.6	s^3	2.5 0.75	[0 0.8 1]
k_3=3.125	s^2	0.8 1	[0 -2.375]
	s	-2.375	
	1		

(d) All coefficients in the table are positive, thus the polynomial is a stable polynomial.

	s^5	1 23 54	
k_1=1/6	s^4	6 52 20	[0 14.3 50.7]
k_2=0.42	s^3	14.3 50.7	[0 30.7 20]
k_3=0.466	s^2	30.7 20	[0 41.38]
	s	41.38	
	1	20	

4. Problem 6.22

$s^3 + a_1 s^2 + a_2 s + a_3$

	s^3	$1 \quad a_2$
$k = 1/a_1$	s^2	$a_1 \quad a_3 \qquad [0 \quad a_2 - a_3/a_1] = [0 \quad \frac{a_1 a_2 - a_3}{a_1}]$
	s	$(a_1 a_2 - a_3)/a_1$
	1	a_3

Stable $\Leftrightarrow$ $a_1 > 0, a_2 > 0, a_3 > 0$ *and* $a_1 a_2 - a_3 > 0.$

$\Leftrightarrow$ $a_1 > 0, a_2 > 0,$ *and* $a_1 a_2 > a_3 > 0.$

5. Problem 6.23

$$H(j\omega) = \frac{j\omega - 2}{-\omega^2 + j\omega + 100.25}$$

$$\omega = 0: H(0) = -2/100.25 = 0.02e^{j\pi} = 0.02e^{j180^\circ}$$

$$\omega = 5: H(5j) = 0.07e^{j1.88} = 0.07e^{j108^\circ}$$

$$\omega = 10: H(10j) = 1.02e^{j0.23} = 1.02e^{j13^\circ}$$

$$\omega = 20: H(20j) = 0.07e^{-j1.4} = 0.07e^{-j80^\circ}$$

$$\omega = 100 : H(100j) \approx \frac{j100}{10000} \approx 0.01e^{-j90^{\circ}}$$

MATLAB code:

```
num = [1 -2]; den = [1 1 100.25]; w=0:0.1:100;
H = freqs(num,den,w);
figure(1); plot(w, abs(H)); xlabel('\omega'); ylabel('Amplitude');
figure(2); plot(w, angle(H)); xlabel('\omega'); ylabel('Angle');
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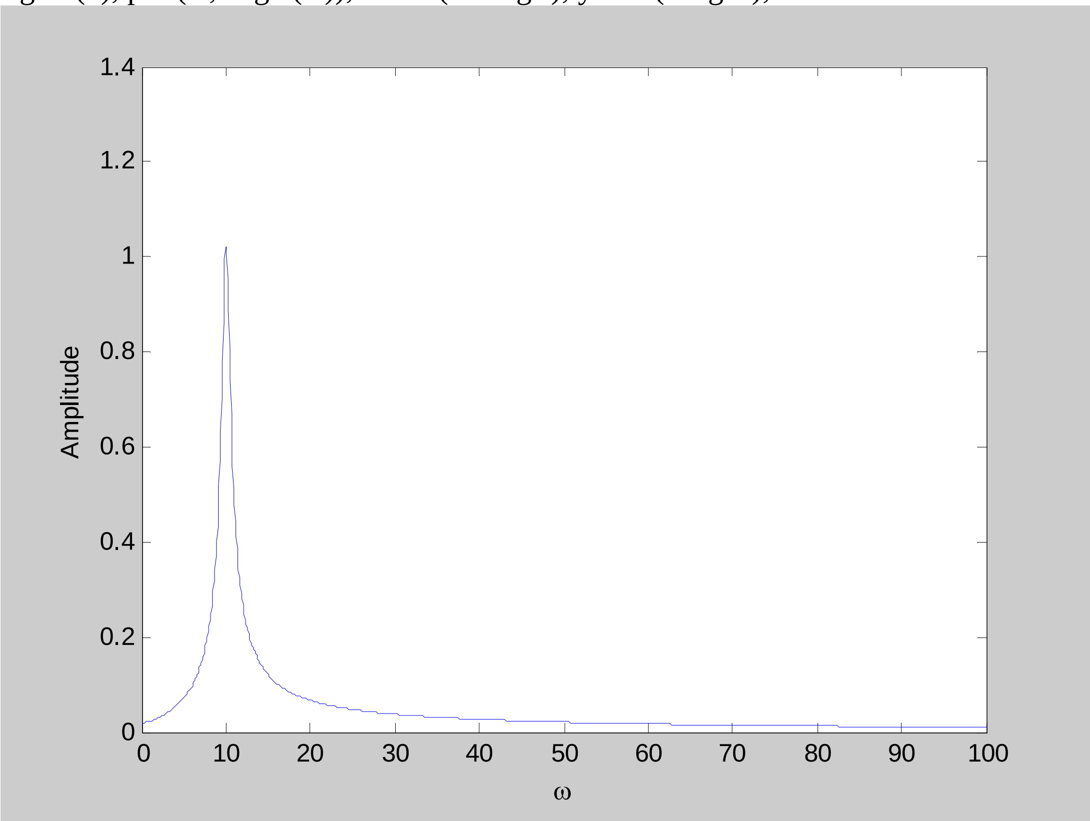
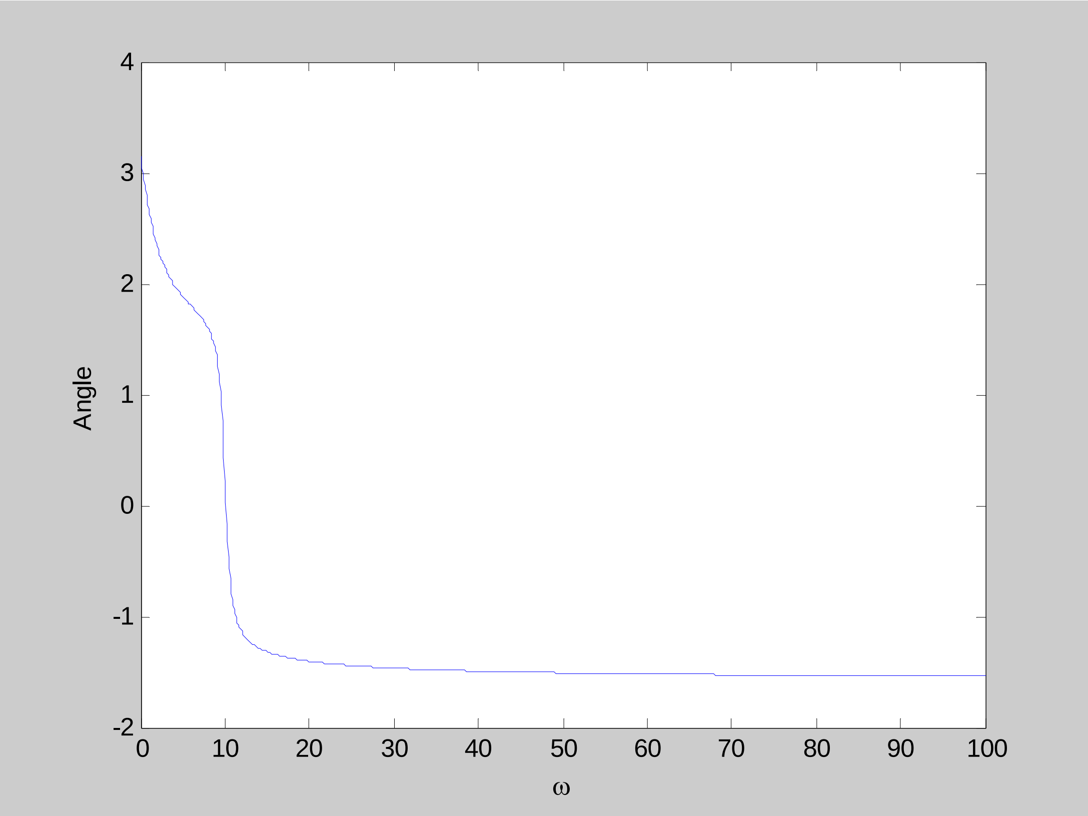
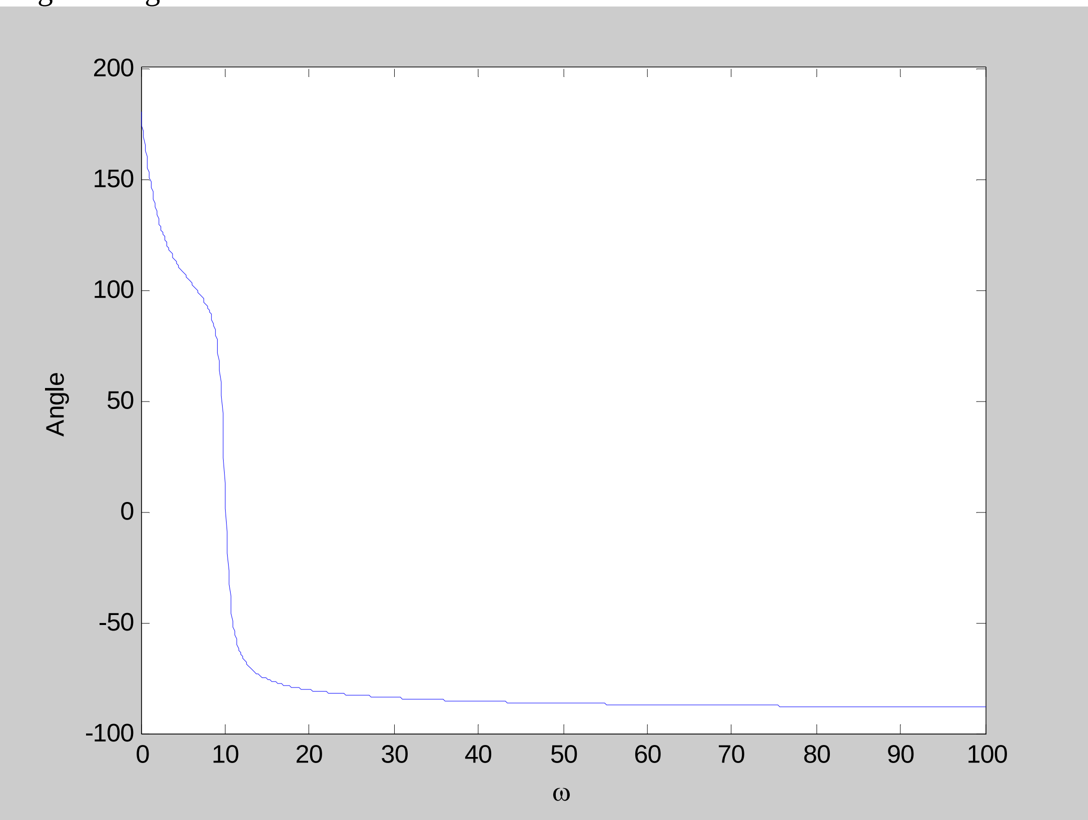


Angle in rad



Angle in degree



6. Problem 6.24

The steady-state response excited by $u(t) = 2 + \sin 5t + 3\cos 10t - 2\sin 100t$ is, using Theorem 6.4

$$y_{ss}(t) = -0.02 \times 2 + 0.07\sin(5t + 1.88) + 1.02 \times 3\cos(10t + 0.23) - 0.01 \times 2\sin(100t - \pi/2)$$
$$\approx 3.06\cos(10t + 0.23)$$

The system has poles $-0.5 \pm j10$. Its time constant is 1/0.5=2. Thus it takes $5 \times 2 = 10s$ to reach steady state. The system is a bandpass filter which passes signals with spectra in the neighborhood of 10 rad/s.

7. Compute the inverse Laplace Transform of the followings:

   a. $X(s) = \dfrac{7s^2 + 15s + 10}{(s+1)^2(s+3)}$

   b. $X(s) = \dfrac{s^4 + 8s^2 + s + 17}{(s^2+4)^2(s+1)}$

(a) Write $X(s) = \frac{7s^2+15s+10}{(s+1)^2(s+3)}$ into partial fraction expansion:

$$X(s) = \frac{a}{(s+1)} + \frac{b}{(s+1)^2} + \frac{c}{(s+3)}$$

Based on Heaviside's theorem:

$$\begin{aligned} a &= \frac{d}{dt}\left((s+1)^2 X(s)\right)|_{s=-1} \\ &= \frac{d}{dt}\left(\frac{7s^2+15s+10}{s+3}\right)|_{s=-1} \\ &= 0 \\ b &= \left((s+1)^2 X(s)\right)|_{s=-1} \\ &= 1 \\ c &= ((s+3)X(s))|_{s=-3} \\ &= 7 \end{aligned}$$

Thus, we have

$$X(s) = \frac{1}{(s+1)^2} + \frac{7}{(s+3)}$$

and the time-domain signal is

$$x(t) = te^{-t}u(t) + 7e^{-3t}u(t)$$

(b) Write $X(s) = \frac{s^4+8s^2+s+17}{(s^2+4)^2(s+1)}$ into partial fraction expansion:

$$X(s) = \frac{a}{s+2i} + \frac{\bar{a}}{s-2i} + \frac{b}{(s+2i)^2} + \frac{\bar{b}}{(s-2i)^2} + \frac{c}{s+1}$$

Applying Heaviside's theorem, we get

$$\begin{aligned} a &= \frac{d}{dt}\left(X(s)(s+2i)^2\right)|_{s=-2i} \\ &= 0.0313i \\ b &= \left(X(s)(s-2i)^2\right)|_{s=2i} \\ &= -0.0625 \\ c &= (X(s)(s+1))|_{s=-1} \\ &= 1 \end{aligned}$$

Applying the inverse Laplace transform, we got

$$x(t) = 0.0313ie^{-2it} - 0.0313ie^{2it} - 0.0625te^{-2it} + 0.0625te^{2it} + e^{-t}$$

You can further simplify the expression to obtain:

$$x(t) = 0.0625\sin(2t) - 0.0125t\cos(2t) + e^{-t}$$

8. You have seen the dramatic collapse of Tacoma Narrows Bridge during lecture. In this problem, we will analyze the cause of the collapse. The left figure below schematically shows the vertical deflection y(t) and the horizontal deflection angle x(t) of the bridge. Of particular importance is the horizontal deflection angle x(t) which is governed by the following differential equation:

$$\frac{d^2}{dt^2}x(t) + c\frac{d}{dt}x(t) + kx(t) = f(t)$$

where c > 0 is the coefficient of viscous damping divided by the mass, k = 1 is the Hooke's law spring constant of the cables divided by the mass, and f(t) is the acceleration of the bridge due to the wind.

a. In the original design of the bridge (the middle figure), the parameter c is found to be very close to 0. Assuming c = 0 and modeling the bridge as H(s) = X(s)/F(s), does it exhibit BIBO stability? Justify your answer.

We use the Laplace transform and get:

$$s^2X(s) - sx(0^-) - x^{(1)}(0^-) + csX(s) - cx(0^-) + kX(s) = F(s) \qquad (5)$$

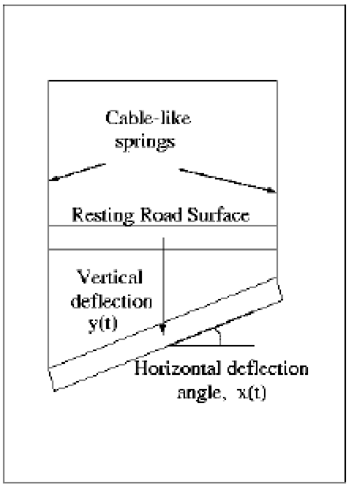
We know $c = 0, k = 1$. To compute the transfer function, we further set all the initial conditions to zero:

$$s^2X(s) + X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2+1}$$

The transfer function $H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2+1}$. The poles are on the imaginary axis: $(0, j), (0, -j)$, so this system is not BIBO. It is a Marginally Stable system.

b. The bridge was later rebuilt with a new design in which engineers replaced the stiffening-plate girders with web trusses as shown in the right figure. This increases the value of c to around two. Comment on the stability of this new design.



In this case, we put $c = 2$ into Eq. 5 and get:

$$s^2X(s) + 2sX(s) + X(s) = F(s)$$

The transfer function $H(s) = \dfrac{X(s)}{F(s)} = \dfrac{1}{(s+1)^2}$. The poles are on the open left half plane and it is BIBO.