

University of Kentucky  
Department of Electrical and Computer Engineering

**EE421G: Signals and Systems I – Fall 2007**

Problem Set 3

Issued: September 10, 2007

Due: September 17, 2008 (In class)

**Reading Assignments:**

Read Chapter 2 (except 2.9) of Chen

**Paper and Pencil Assignments:**

1) Problem 1.31: Consider  $\cos \omega_k n$  with  $T=1$  and

$$\omega_k = \frac{2\pi k}{N}$$

where  $N$  is a given positive integer and  $k$  is an integer ranging from  $-\infty$  to  $\infty$ . How many different  $\cos \omega_k n$  are there (in terms of  $N$ )? What are their frequencies (in terms of  $N$ )?

Solution:

For  $T=1$ , the Nyquist frequency range (NFR) is  $(-\pi, \pi]$ . There are  $N$  different  $\cos \omega_k n$ , because

$$\begin{aligned}\omega_{k+N} &= \frac{2\pi(k+N)}{N} = \frac{2\pi k}{N} + 2\pi \\ &= \frac{2\pi k}{N} = \omega_k \pmod{2\pi}\end{aligned}$$

For  $N$  even,  $\omega_k$  with  $k = -\frac{N}{2} + 1 : \frac{N}{2}$  are inside  $(-\pi, \pi]$ .

For  $N$  odd,  $\omega_k$  with  $k = -\frac{N-1}{2} : \frac{N-1}{2}$  are inside  $(-\pi, \pi]$ .

For example, for  $N=6$ , the six  $k$ 's are

$$k = -\frac{6}{2} + 1 : \frac{6}{2} = -2 : 3 = -2, -1, 0, 1, 2, 3$$

For  $N=7$ , the seven  $k$ 's are

$$k = -\frac{7-1}{2} : \frac{7-1}{2} = -3 : 3 = -3, -2, -1, 0, 1, 2, 3$$

2) Problem 1.35: What are the sample sequences of  $x(t) = \cos 50t + 2 \sin 70t$  for  $T = \pi/45$ ,  $\pi/60$  and  $\pi/180$ . Under what condition on  $T$  will all frequencies of  $x(t)$  be retained in  $x(nT)$ ?

Solution:

$$x(nT) = \cos(50nT) + 2 \sin(70nT)$$

$$T = \pi/45: \text{NFR} = (-45, 45], x(nT) = \cos(40nT) - 2 \sin(20nT)$$

which does not reveal both frequencies of  $x(t)$ .

$$T = \pi/60: \text{NFR} = (-60, 60], x(nT) = \cos(50nT) - 2 \sin(50nT)$$

which does not reveal the frequency 70 rad/s of  $x(t)$ .

$$T = \pi/180: \text{NFR} = (-180, 180], x(nT) = \cos(50nT) + 2 \sin(70nT)$$

which reveals both frequencies of  $x(t)$ .

If  $T < \frac{\pi}{70}$  or  $f_s > \frac{70}{\pi}$  Hz or  $\omega_s = 2\pi f_s > 140$  rad/s,

then all frequencies of  $x(t)$  will be retained in  $x(nT)$ .

3) Problem 2.3: Consider a CT system whose input and output are related by

$$y(t) = \begin{cases} u^2(t)/u(t-1) & \text{if } u(t-1) \neq 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

Show that the system satisfies the homogeneity property but not the additivity property.

Thus the homogeneity property does not imply the additivity property.

Solution:

$$\text{If } u_1(t) = \alpha u(t)$$

$$y_1(t) = \frac{\alpha^2 u^2(t)}{\alpha u(t-1)} = \alpha \frac{u^2(t)}{u(t-1)} = \alpha y(t)$$

Thus the homogeneity property holds.

If  $u(t) = u_1(t) + u_2(t)$ , then

$$\frac{[u_1(t) + u_2(t)]^2}{u_1(t-1) + u_2(t-1)} \neq \frac{u_1^2(t)}{u_1(t-1)} + \frac{u_2^2(t)}{u_2(t-1)}$$

Thus the additivity property does not hold.

So the homogeneity property does not imply the additivity property.

4) Problem 2.4: Show that if  $\{u_1 + u_2\} \rightarrow \{y_1 + y_2\}$ , then

$$\left\{ \frac{a}{b} u_1 \right\} \rightarrow \left\{ \frac{a}{b} y_1 \right\}$$

for any integers  $a$  and  $b$ . Thus the additivity property implies  $\{\alpha u_1\} \rightarrow \{\alpha y_1\}$  for any rational number  $\alpha$ . In other words, the additivity property *almost* implies the homogeneity property.

Solution:

We use  $y = L[u]$  to denote  $u \rightarrow y$ . Let  $a$  be a positive integer, then

$$\begin{aligned} L[au] &= L[u + (a-1)u] = L[u] + L[(a-1)u] \\ &= L[u] + L[u + (a-2)u] = L[u] + L[u] + L[(a-2)u] \\ &= \dots = L[u] + L[u] + \dots L[u] = aL[u] \\ L[u+0] &= L[u] + L[0] = L[u] \Rightarrow L[0] = 0 \\ L[au + (-au)] &= L[au] + L[-au] = L[0] = 0 \\ L[-au] &= -L[au] = -aL[u] \end{aligned}$$

Thus additivity implies  $L[au] = aL[u]$  for any positive or negative integer.

Let  $\alpha = a/b$  for any integers  $a$  and  $b$ ,

Define  $u = bv$  then

$$L[u] = L[bv] = bL[v]$$

$$\Rightarrow L[v] = \frac{1}{b}L[u]$$

$$\text{or } L\left[\frac{u}{b}\right] = \frac{1}{b}L[u]$$

Now

$$L[\alpha u] = L\left[\frac{a}{b}u\right] = aL\left[\frac{u}{b}\right] = \frac{a}{b}L[u]$$

Thus additivity implies

$$L[\alpha u] = \alpha L[u] \text{ for any rational number } \alpha .$$

5) Problem 2.5: Discuss whether or not each of the following equations is memoryless, linear, time-invariant, and causal:

a.  $y(t) = -2 + 3u(t)$

b.  $y(t) = \sqrt{u(t)}$

c.  $y(t) = u(t)u(t-1)$

d.  $y(t) = tu(t)$

e.  $y(t) = \int_{t_0}^t u(\tau) d\tau + y(t_0)$

f.  $y(t) = \int_{t_0}^t tu(\tau) d\tau + y(t_0)$

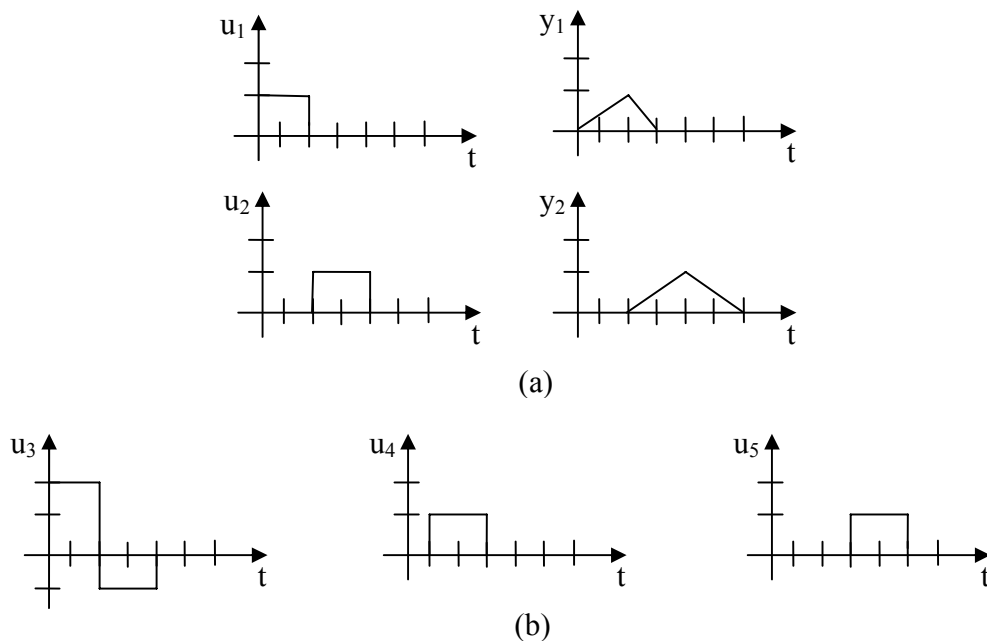
Solution:

TI: time-invariant

- memoryless, nonlinear, TI, causal
- memoryless, nonlinear, TI, causal
- with memory, nonlinear, TI, causal
- memoryless, linear, time-varying, causal
- with memory, linear, TI, causal

f. with memory, linear, time-varying, causal

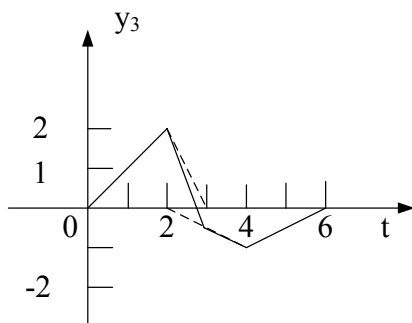
6) Problem 2.7: Consider a CT linear system. Its zero-state responses excited by  $u_1$  and  $u_2$  are shown in the following figure (a). Is the system time-varying or time-invariant? Can you find the zero-state responses excited by the inputs  $u_3$ ,  $u_4$ , and  $u_5$  shown in figure (b)?



Solution:

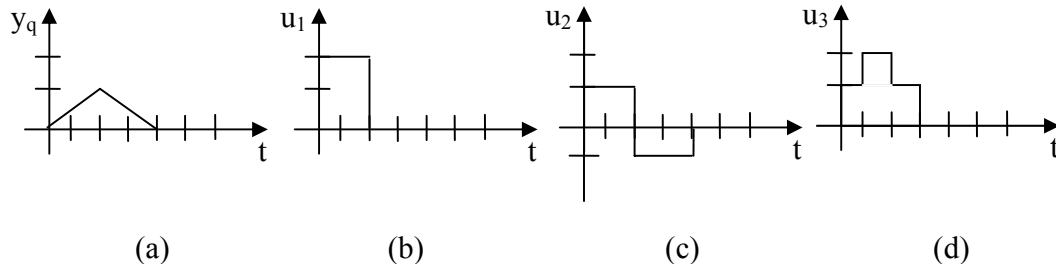
It's time varying, thus the time shifting property cannot be used.

$$u_3 = 2u_1 - u_2 \Rightarrow y_3 = 2y_1 - y_2$$



Here we have used only homogeneity and additivity properties. Although  $u_4$  is the shifting of  $u_1$  or  $u_2$  by one second,  $y_4$  is not necessarily the shifting of  $y_1$  or  $y_2$  by one second. Thus it is not possible to compute  $y_4$  from the given pairs. The same conclusion applies to  $y_5$ .

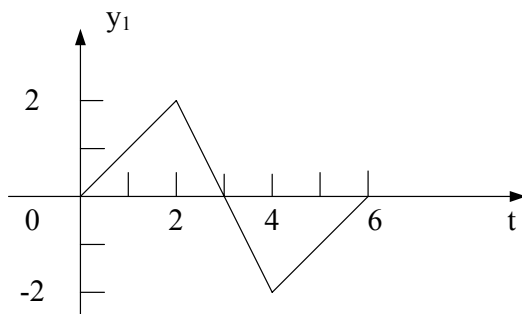
- 7) Problem 2.9: Consider a CT LTI system. Suppose its step response (the zero-state response excited by a step input or  $u(t) = q(t)$  is as shown in Figure (a). Find the outputs excited by the inputs shown in Figure (b) through (d).



Solution:

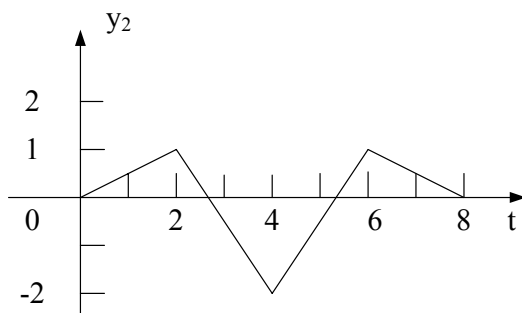
$$u_1(t) = 2q(t) - 2q(t-2)$$

$$\Rightarrow y_1(t) = 2y_q(t) - 2y_q(t-2)$$



$$u_2(t) = 0.5u_1(t) - 0.5u_1(t-2)$$

$$\Rightarrow y_2(t) = 0.5y_1(t) - 0.5y_1(t-2)$$



$$u_3(t) = 0.5u_1(t) + 0.5u_1(t-1)$$

$$\Rightarrow y_3(t) = 0.5y_1(t) + 0.5y_1(t-1)$$

