

University of Kentucky  
Department of Electrical and Computer Engineering

**EE421G: Signals and Systems I – Fall 2007**

Problem Set 7

Issued: October 13, 2007

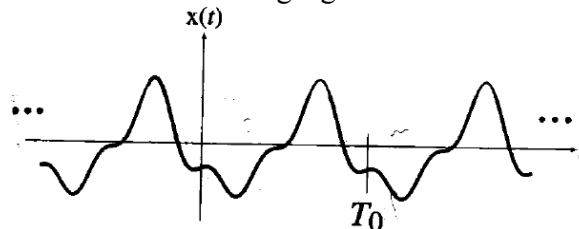
Due: October 19, 2007 (In class)

**Reading Assignments:**

Read Chapter 4 of Chen

**Computer Assignment:**

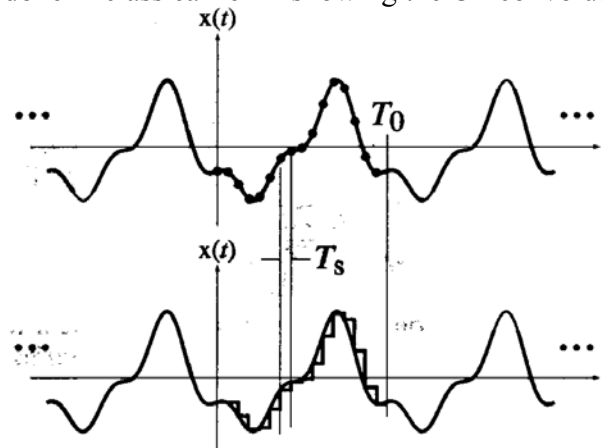
In this problem, we learn how to numerically compute the Fourier series of an arbitrary periodic signal. Let's consider the following signal:



This signal presents some problems. It is not at all obvious how to describe it. Up to this time in our study, we needed a mathematical description of a signal so that we can symbolically compute its Fourier series coefficient:

$$c_k = \frac{1}{T_0} \int_0^{T_0} x(t) \exp(-jk\omega_0 t) dt$$

There is a better way – if we have a set of N samples of the signal taken from one period, we can estimate  $c_k$  numerically. The larger the N, the better the approximation becomes. Let's say, the sampling interval is  $T_s$ . We can approximate the signal as a piece-wise flat function as we have done in class earlier in showing the CT convolution formula.



The Fourier series coefficient can then be approximated as a sum of the area of rectangles:

$$\begin{aligned} c_k &\approx \frac{1}{T_0} \sum_{n=0}^{N-1} x(nT_s) \exp(-jk\omega_0 nT_s) T_s \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x(nT_s) \exp\left(-j \frac{k2\pi}{N} n\right) \end{aligned}$$

The last line uses the simple relationship  $T_0 = NT_s$  (N samples in one period) and  $\omega_0 = 2\pi/T_0 = 2\pi/(NT_s)$ . The summation  $\sum_{n=0}^{N-1} x(nT_s) \exp\left(-j \frac{k2\pi}{N} n\right)$  turns out to be a very

important Fourier representation called Discrete Fourier Transform (DFT), and it can be efficiently computed using Fast Fourier Transform (FFT). Thus, if  $x$  is a N-sample version of  $x(t)$  in MATLAB, we can use the following to compute the Fourier series coefficient:

```
>> c = fft(x)/length(N)
```

As it turns out, only the first half of the vector  $c$  represents the first  $N/2$  Fourier series coefficients of  $x(t)$ . The remaining half represents a reflected version of the conjugate of the same set of coefficients which we can ignore.

Okay, it's your turn to do something. Use the above method to find the magnitude and phase spectrums of the first 64 Fourier series coefficients of the periodic signal  $x(t)$ , one period of which is described by

$$x(t) = \sqrt{1-t^2} \quad -1 < t < 1$$

Here are some hints to get you started. The fundamental period of the signal is 2. Use  $N=128$  to sample the signal between 0 and 2. To do that, we first define the time instances to do the sampling:

```
>> n = 0:127;
>> Ts = 2/128;
>> t = Ts*n;
```

Then, we can do the sampling.

```
>> x = [sqrt(1-t(1:64).^2) sqrt(1-(t(65:128)-2).^2)]; % Can you explain?
```

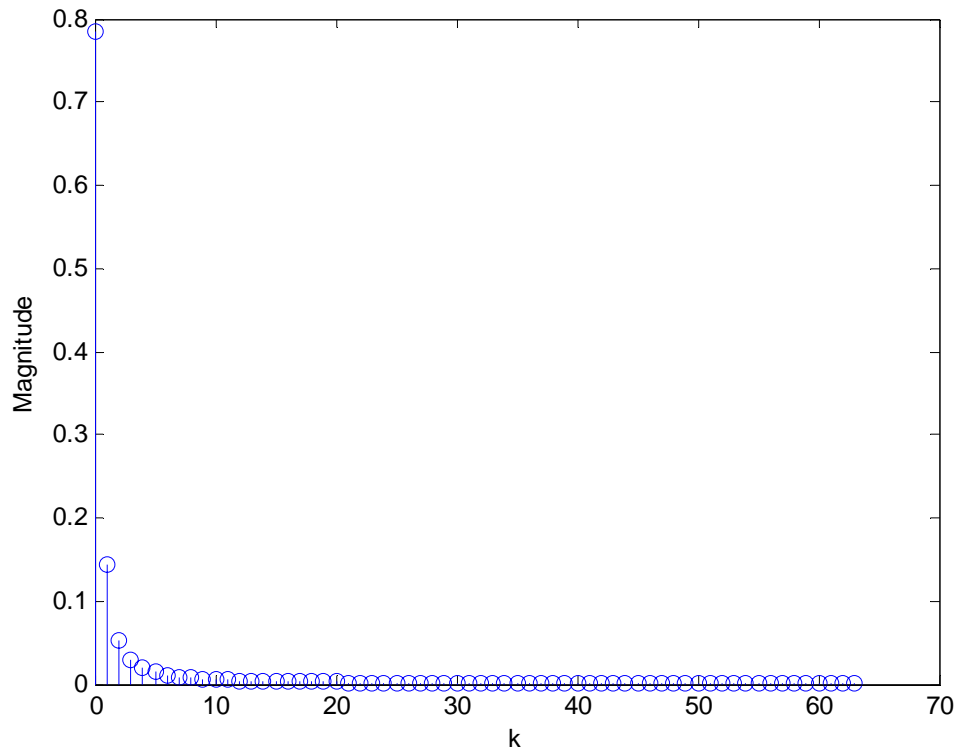
Finally, apply the FFT and use the stem plots to show the magnitude (with command `abs`) and phase (with command `angle`) of the harmonics.

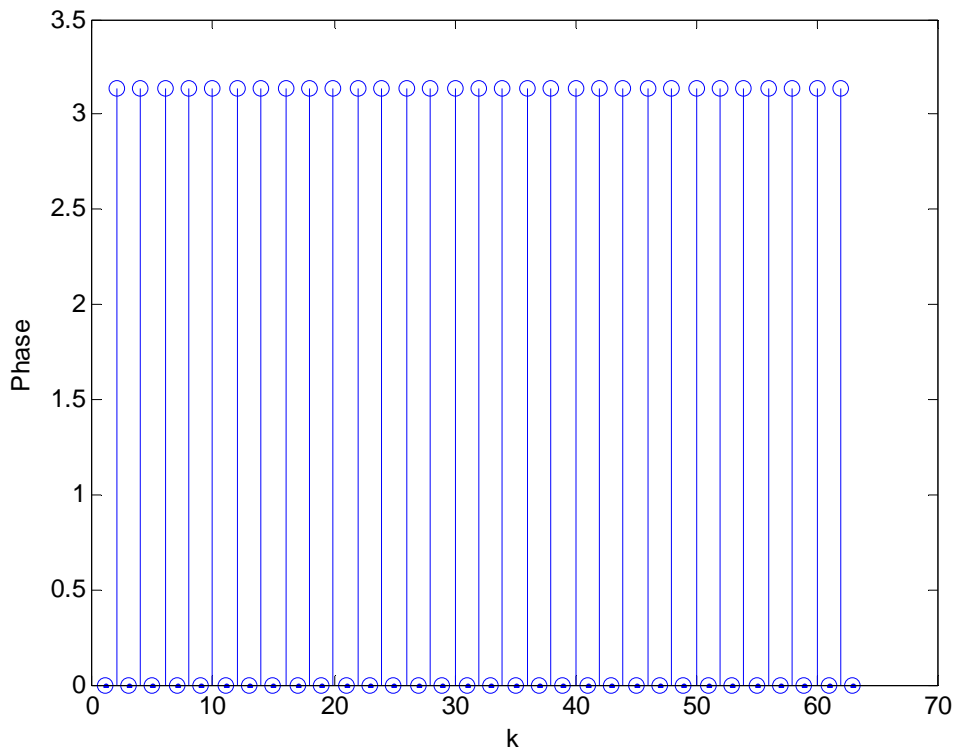
Solution:

MATLAB code:

```
clear;
close all;
n = 0:127;
Ts = 2/128;
t = Ts*n;
x = [sqrt(1-t(1:64).^2) sqrt(1-(t(65:128)-2).^2)];
figure;
plot(t, x);
c = fft(x)/length(n);
figure;
```

```
stem(n(1:length(n)/2), abs(c(1:length(n)/2)));  
xlabel('k');  
ylabel('Magnitude');  
figure;  
stem(n(1:length(n)/2), angle(c(1:length(n)/2)));  
xlabel('k');  
ylabel('Phase');
```





**Paper and Pencil Assignments:**

1) Problem 4.9

Solution:

Let  $c_m$  be the Fourier series coefficient using  $\omega_0 = 2\pi / P$  and  $\bar{c}_m$  be the coefficient using  $\omega_1 = 2\pi / 2P = \omega_0 / 2$ . Then

$$\bar{c}_m = 0 \text{ for all } m \text{ odd}$$

$$\bar{c}_m = c_{(m/2)} \text{ for all } m \text{ even}$$

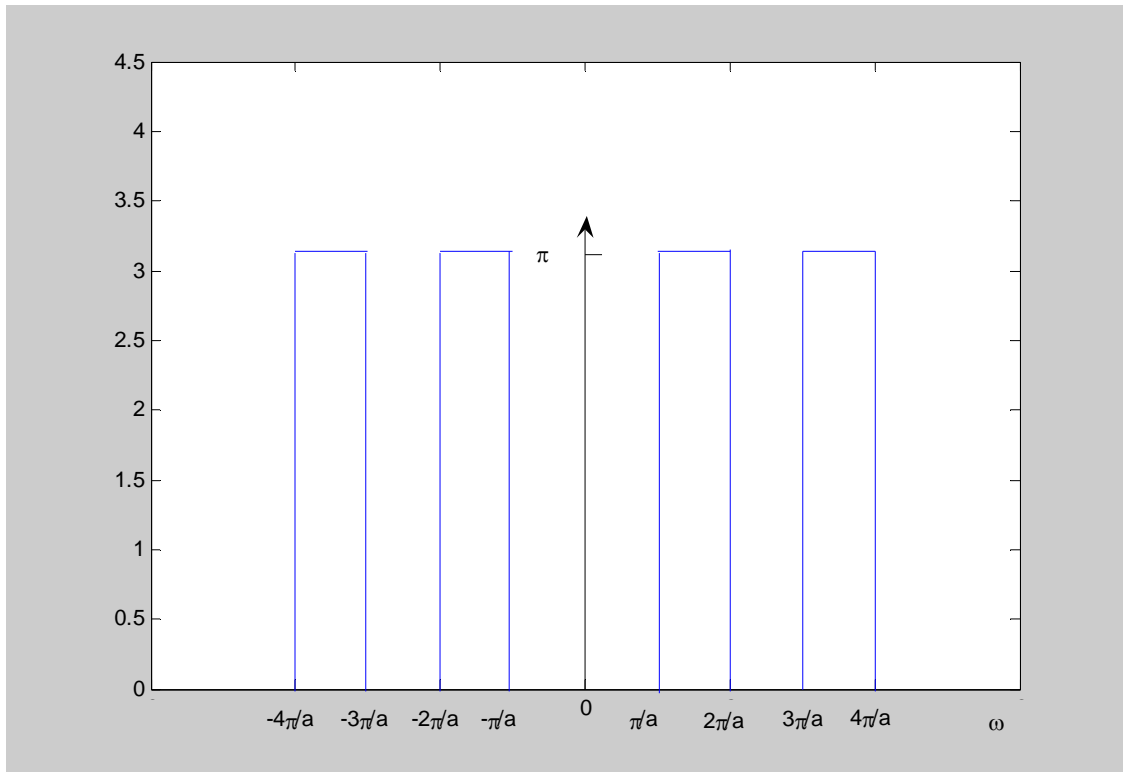
If we do not use the fundamental period, then it requires more computation. However, the final result is still the same.

2) Problem 4.10

Solution:

The spectrum of the window is  $W_a(\omega) = \frac{2 \sin a\omega}{\omega}$ . Its magnitude spectrum is as shown in

Figure 4.7 (c). Its phase is 0 if  $W_a(\omega) > 0$ ,  $\pi$  if  $W_a(\omega) < 0$ . Thus from Fig. 4.7(b), we can obtain its phase spectrum as



Note that if  $W_a(\omega) < 0$ , its phase can be selected as  $\pi$  or  $-\pi$ . If we select  $\pi$  for  $\omega > 0$  and  $-\pi$  for  $\omega < 0$ , then the phase plot is odd in the usual sense.

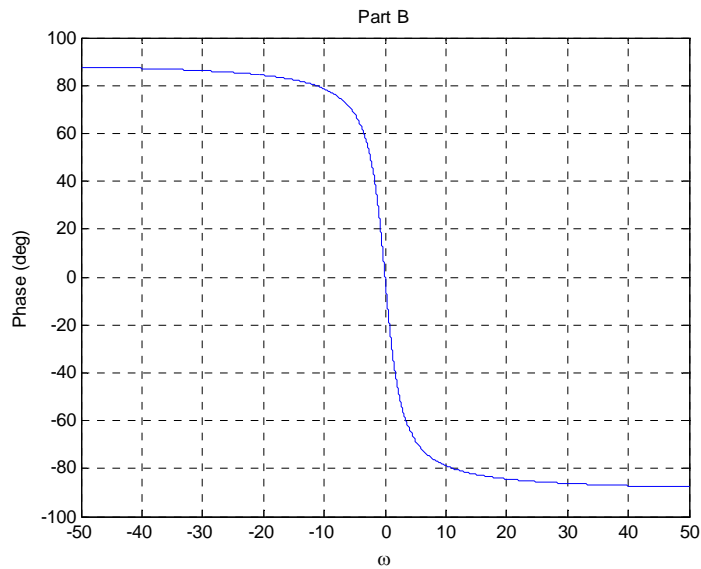
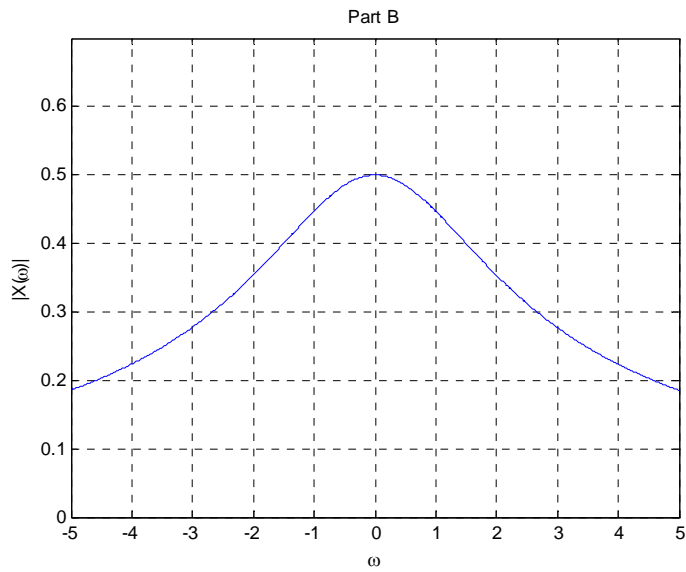
### 3) Problem 4.11

Solution:

(a)  $e^{-2t} \rightarrow \infty$  as  $t \rightarrow -\infty$ , thus its Fourier transform diverges and its spectrum is not defined.

(b)  $x(t) = e^{-2t}$ , for  $t \geq 0$

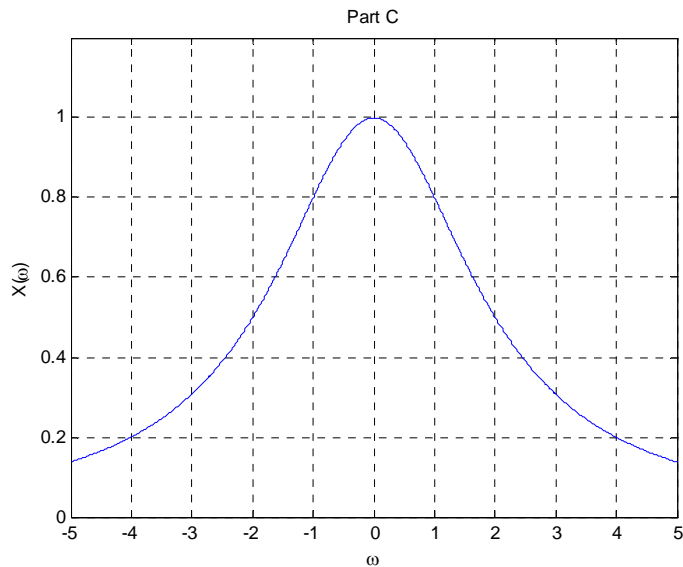
$$\begin{aligned}
 X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_0^{\infty} e^{-(j\omega+2)t} dt \\
 &= \frac{1}{-(j\omega+2)} e^{-(j\omega+2)t} \Bigg|_{t=0}^{\infty} = \frac{1}{-(j\omega+2)} (0-1) \\
 &= \frac{1}{j\omega+2}
 \end{aligned}$$



$$(c) x(t) = \begin{cases} e^{-2t}, & t \geq 0 \\ e^{2t}, & t < 0 \end{cases}$$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^0 e^{2t} e^{-j\omega t} dt + \int_0^{\infty} e^{-2t} e^{-j\omega t} dt \\ &= \frac{1}{2-j\omega} e^{(2-j\omega)t} \Big|_{t=-\infty}^0 + \frac{-1}{2+j\omega} e^{-(2+j\omega)t} \Big|_{t=0}^{\infty} \\ &= \frac{1}{2-j\omega} (1-0) + \frac{-1}{2+j\omega} (0-1) \\ &= \frac{1}{2-j\omega} + \frac{1}{2+j\omega} = \frac{4}{4-(j\omega)^2} = \frac{4}{4+\omega^2} \end{aligned}$$

$X(\omega)$  is real and positive. Its phase is 0 for all  $\omega$ .



$$(d) x(t) = \begin{cases} e^{-3t}, & t \geq 0 \\ 2e^{4t}, & t < 0 \end{cases}$$

$$X(\omega) = \frac{1}{j\omega + 3} + \frac{2}{-j\omega + 4} = \frac{j\omega + 10}{(j\omega + 3)(-j\omega + 4)}$$

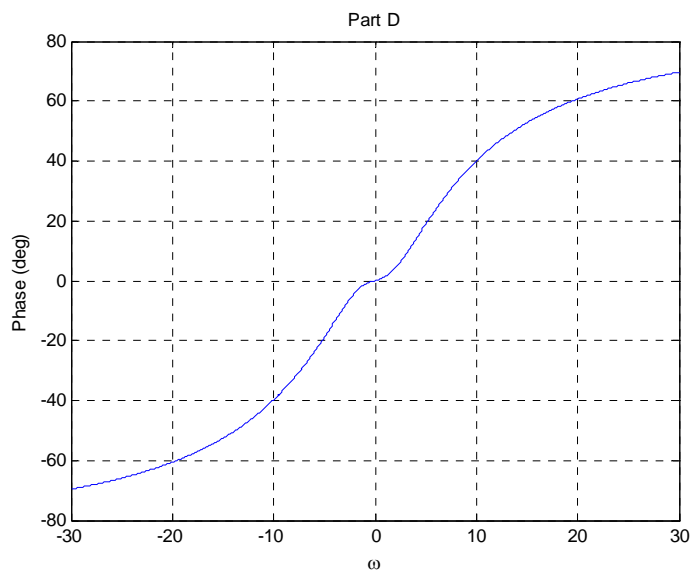
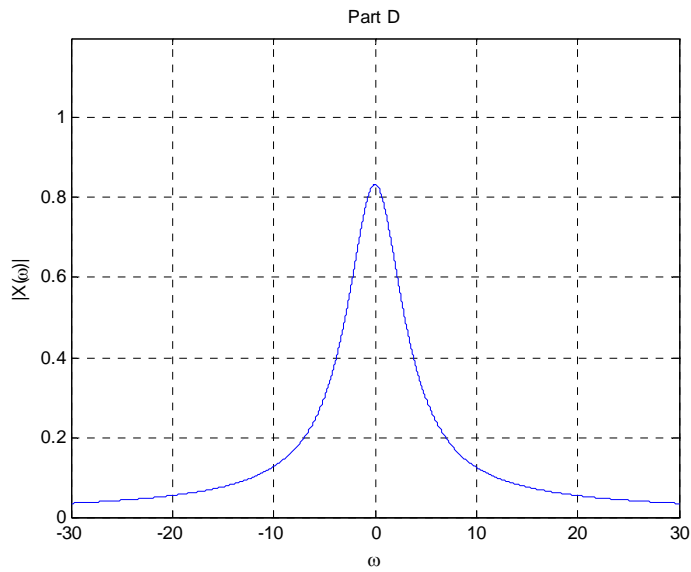
$$= \frac{j\omega + 10}{\omega^2 + j\omega + 12}$$

$$X(0) = \frac{10}{12} = 0.83e^{j0^\circ}$$

$$\omega \rightarrow \infty, X(\omega) \approx \frac{j\omega}{\omega^2} = \frac{j}{\omega}, |X(\omega)| \rightarrow 0, \angle X(\omega) \rightarrow 90^\circ$$

$$X(5) = 0.3e^{j19^\circ}$$

$$X(30) = 0.035e^{j69.7^\circ}$$



4) Problem 4.12

Solution:

If  $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$ , then

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

This becomes, after interchanging  $\omega$  and  $t$ ,

$$x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t)e^{jt\omega} dt$$

and replacing  $\omega$  by  $-\omega$ ,

$$x(-\omega) = \frac{1}{2\pi} \int_{t=-\infty}^{\infty} X(t) e^{-j\omega t} dt = \frac{1}{2\pi} \mathcal{F}[X(t)]$$

$$\text{From (4.31) and (4.32), we have } W_a(\omega) = \frac{2 \sin a\omega}{\omega} = \mathcal{F}[w_a(t)]$$

The duality property implies

$$W_a(-\omega) = \mathcal{F}\left[\frac{1}{2\pi} \cdot \frac{2 \sin at}{t}\right] = \mathcal{F}\left[\frac{\sin at}{\pi t}\right]$$

Using  $w_a(-\omega) = w_a(\omega)$  (even)

and replacing  $a$  by  $\omega_c$ , we obtain (4.33).

### 5) Problem 4.15

Solution:

$$\bar{X}(\omega) = \mathcal{F}[e^{-0.1t}, t \geq 0] = \frac{1}{j\omega + 0.1}$$

$$\mathcal{F}[\bar{x}(t) \sin \omega_0 t] = -0.5j(\bar{X}(\omega - \omega_0) - \bar{X}(\omega + \omega_0))$$

$$x(t) = e^{-0.1t} \sin(10t + \pi/4)$$

$$= e^{-0.1t} \left[ \sin 10t \cdot \cos \frac{\pi}{4} + \cos 10t \cdot \sin \frac{\pi}{4} \right]$$

$$= 0.7[e^{-0.1t} \sin 10t + e^{-0.1t} \cos 10t]$$

$$X(\omega) = \mathcal{F}[x(t)] = 0.7 \left\{ -0.5j \left[ \frac{1}{j(\omega - 10) + 0.1} - \frac{1}{j(\omega + 10) + 0.1} \right] \right. \\ \left. + 0.5 \left[ \frac{1}{j(\omega - 10) + 0.1} + \frac{1}{j(\omega + 10) + 0.1} \right] \right\}$$

$$= 0.7 \left\{ \frac{0.5 - j0.5}{j(\omega - 10) + 0.1} + \frac{0.5 + j0.5}{j(\omega + 10) + 0.1} \right\}$$

$$= 0.7 \frac{j\omega + 10.1}{-\omega^2 + j0.2\omega + 100.01}$$

### 6) Problem 4.16

Solution:

We use (4.49) to write

$$LB = \frac{\left( \int_{-\infty}^{\infty} |x(t)| dt \right)^2}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \cdot \frac{\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}{2|X(0)|^2}$$

$$= \frac{\pi \left( \int_{-\infty}^{\infty} |x(t)| dt \right)^2}{|X(0)|^2}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

$$|X(0)| \leq \int_{-\infty}^{\infty} |x(t)| dt$$

Thus we have

$$LB = \pi \left[ \frac{\int_{-\infty}^{\infty} |x(t)| dt}{|X(0)|} \right]^2 \geq \pi$$

7) Problem 4.19

Solution:

$$X_m(\omega) = 0.5(X(\omega - \omega_c) + X(\omega + \omega_c))$$

First we have  $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(\omega - a)|^2 d\omega$  for any real  $a$ .

If  $X(\omega)$  is bandlimited to  $W$  and if  $\omega_c > 2W$ , there is no overlapping of  $X(\omega - \omega_c)$  and

$X(\omega + \omega_c)$ . Thus we have  $|X_m(\omega)| = 0.5|X(\omega - \omega_c)| + 0.5|X(\omega + \omega_c)|$

and

$$\begin{aligned} \int_{-\infty}^{\infty} |X_m(\omega)|^2 d\omega &= 0.25 \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega + 0.25 \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \\ &= 0.5 \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \end{aligned}$$

Thus the total energy of  $x_m(t)$  is half of the total energy of  $x(t)$ . If  $\omega_c < 2W$ ,  $X(\omega - \omega_c)$  and  $X(\omega + \omega_c)$  overlap and the preceding equation does not hold and the assertion does not hold.