Sample Midterm

1. A continuous-time signal x(t) is sampled with sampling period T=0.5 second.

   a. What is the period (in radian/second) of its Fourier Transform?

   The period is just the sampling frequency $\omega_s = 2\pi/T = 4\pi = 12.57$ rad/s

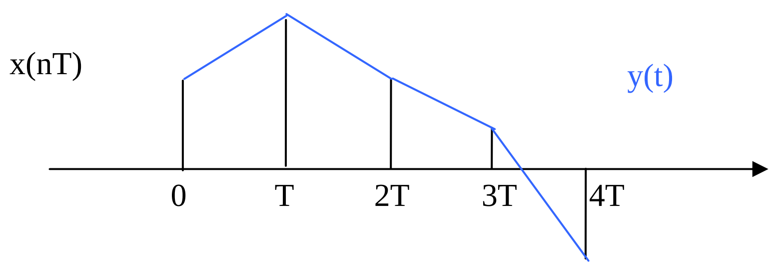
   b. If the highest frequency component of x(t) is 5 Hz, can we fully recover the original continuous-time signal based on the discrete-time sample x(nT)? Justify your answer.

   Since the sampling frequency is 1/0.5 or 2 Hz, the highest frequency that can pass through without aliasing is 2/2 = 1 Hz according to the Nyquist Theorem. Thus we cannot recover the original signal.

   c. Give three different types of D/A methods and describe their relative performance in terms of complexity and reconstruction fidelity.

   In decreasing order of complexity and reconstruction accuracy:
   Ideal LowPass > RC filter > Linear Interpolation > Sample-and-Hold

   d. Given the discrete-time input x(nT) to a LINEAR INTERPOLATION D/A converter, sketch the output reconstructed continuous signal y(t) on top of x(nT).



2. Sampling and Interpolation

   a. (4 points) State the Nyquist Theorem

   Nyquist Theorem states that a continuous-time band-limited signal can be perfectly reconstructed from its discrete samples if the sampling frequency is twice the bandwidth.

   b. (7 points) Explain why linear interpolation is a better D/A technique than sample-and-hold.

   Linear interpolation is better than sample-and-hold as the frequency response of linear interpolation decays faster than sample-and-hold and thus better suppress the high frequency components introduced by sampling.

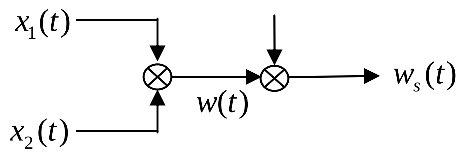
   c. (7 points) A continuous-time signal $x_c(t)=\cos(4000\pi t)$ is sampled with sampling period T to obtain the discrete-time signal $x_c(nT) = x_d(n) = \cos(\pi n/3)$. Determine a choice of T that is consistent with this information.

Since $x_d(n) = x_c(nT)$, we have $4000\pi nT = \pi n/3 \Rightarrow T = \frac{1}{12000}$

d. (7 points) Continued from part c., is your choice for T unique? If so, explain why. If not, specify another choice of T.

T is not unique because $\cos(4000\pi nT) = \cos(\frac{\pi n}{3} + 2m\pi)$ for any integer m. In particular we can choose m = n, and have $4000\pi nT = \pi n/3 + 2n\pi \Rightarrow T = \frac{7}{12000}$

3. In the following system, two continuous-time functions $x_1(t)$ and $x_2(t)$ are MULTIPLIED and the product $w(t)$ is sampled with sampling period T.



If the spectrums of $x_1(t)$ and $x_2(t)$ are given as follows, determine the maximum sampling period T such that $w(t)$ is recoverable from $w_s(t)$ through the use of an ideal lowpass filter.

Since multiplication in time domain is equivalent to convolution in frequency domain, the signal $w(t) = x_1(t)x_2(t)$ is bandlimited within [-10-5 Hz,10+5 hz] or [-15 Hz, 15Hz]. Thus the maximum sampling period T = 1/(15·2) = 1/30 seconds.

4. Discrete-Time Fourier Transform

a) Given the DTFT of a LTI system is $H_d(\omega) = \sum_{n=0}^{\infty} h(nT)e^{-j\omega Tn}$, show that if the impulse response h(nT) is real, we have $H_d(-\omega) = \overline{H_d(\omega)}$, i.e. the negative frequency content can be deduced by taking the conjugate of the positive frequency content.

We can go straight from the definition:

$$H(-\omega) = \sum_{n=0}^{\infty} h(nT)e^{j\omega Tn}$$

$$= \overline{\sum_{n=0}^{\infty} h(nT)e^{-j\omega Tn}} = \overline{H(\omega)}$$

b) If the input to $H(\omega)$ is $x(nT) = \cos(\omega_0 nT) = \frac{1}{2}(e^{j\omega_0 nT} + e^{-j\omega_0 nT})$, use part a) to show that the output is given by

$$y(nT) = |H(e^{j\omega_0 T})|\cos\left(\omega_0 nT + \angle H(e^{j\omega_0 T})\right)$$

As $x(nT) = \cos(\omega_0 nT) = \frac{1}{2}(e^{j\omega_0 nT} + e^{-j\omega_0 nT})$ and we know that the complex exponential is the "eigen-signal" of any LTI system. Thus, the output

$$\begin{aligned} y(nT) &= \tfrac{1}{2}\left(H(\omega_0)e^{j\omega_0 nT} + \overline{H(\omega_0)}e^{-j\omega_0 nT}\right) \\ &= |H(\omega_0)|\cdot\tfrac{1}{2}\left(\exp\left(j(\omega_0 nT + \angle H(\omega_0))\right) + \exp\left(j(-\omega_0 nT - \angle H(\omega_0))\right)\right) \\ &= |H(\omega_0)|\cos(\omega_0 nT + \angle H(\omega_0)) \end{aligned}$$

5. Laplace Transform

a. No need to compute the actual coefficients, write down the INVERSE LAPLACE TRANSFORM of the followings:

i) $X(s) = \dfrac{7s^3 + 20s^2 + 33s + 82}{(s^2+4)(s+2)(s+3)}$

$$X(s) = \frac{A}{s-j2} + \frac{\overline{A}}{s+j2} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$x(t) = Ae^{j2t} + \overline{A}e^{-j2t} + Be^{-2t} + Ce^{-3t}$$

ii) $X(s) = \dfrac{s^2(s+9)}{(s+3)^3(s+1)}$

$$X(s) = \frac{A}{(s+3)^3} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)} + \frac{D}{s+1}$$

$$x(t) = \frac{A}{2}t^2e^{-3t} + Bte^{-3t} + Ce^{-3t} + De^{-t}$$

b. A dynamic system is governed by the following differential equation with initial conditions $\left.\dfrac{dy}{dt}\right|_{t=0} = 1, y(0) = 0$

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = x(t)$$

Given the input X(s) = 1. Write down the Laplace transform of the zero-state response and the zero-input response.

Writing the differential equation in complex domain, we have:

$$s^2Y(s) - sy(0) - y^{(1)}(0) + 2sY(s) - 2y(0) + Y(s) = X(s)$$
$$(s^2 + 2s + 1)Y(s) = sy(0) + y^{(1)}(0) + 2y(0) + X(s)$$
$$Y(s) = \frac{sy(0) + y^{(1)}(0) + 2y(0)}{s^2 + 2s + 1} + \frac{X(s)}{s^2 + 2s + 1}$$

The first term depends only on the initial condition so it is the zero-input response. Putting in the numerical value we have:

$$Y_{ZIR}(s) = \frac{1}{s^2 + 2s + 1}$$

The second term depends only on the input so it is the zero-state response. Putting in the numerical value we have:

$$Y_{ZSR}(s) = \frac{1}{s^2 + 2s + 1}$$

6. Discrete Fourier Transform

Let x(n) and y(n) be two three-point sequence:

$$x(n) = \begin{cases} 1 & \text{for n} = 0 \\ 2 & \text{for n} = 1 \\ 1 & \text{for n} = 2 \end{cases} \qquad y(n) = \begin{cases} -1 & \text{for n} = 0 \\ 2 & \text{for n} = 1 \\ 1 & \text{for n} = 2 \end{cases}$$

Compute the *5-point* DFT X(k) for x(n). You do not need to simplify your answers.

$$X(0) = 1 + 2 + 1 = 4$$
$$X(1) = 1 + 2\cdot\exp\left(-j\frac{2\pi}{5}\right) + \exp\left(-j\frac{4\pi}{5}\right)$$
$$X(2) = 1 + 2\cdot\exp\left(-j\frac{4\pi}{5}\right) + \exp\left(-j\frac{8\pi}{5}\right)$$
$$X(3) = 1 + 2\cdot\exp\left(-j\frac{6\pi}{5}\right) + \exp\left(-j\frac{12\pi}{5}\right)$$
$$X(4) = 1 + 2\cdot\exp\left(-j\frac{8\pi}{5}\right) + \exp\left(-j\frac{16\pi}{5}\right)$$

7. Assume $x(t) = a_1(t)x_1(t) + a_2(t)x_2(t)$, $X_1(s) = L[x_1(t)]$ and $X_2(s) = L[x_2(t)]$.
   a. Is $X(s) = L[x(t)]$ equal to $a_1(t)X_1(s) + a_2(t)X_2(s)$ and why?

   No. The Laplace transform is a function of s not t, and thus cannot contain terms like $a_1(t)$ and $a_2(t)$. – 2 points

   b. If $A_1(s) = L[a_1(t)]$ and $A_2(s) = L[a_2(t)]$, is $L[x(t)] = A_1(s)X_1(s) + A_2(s)X_2(s)$ and why?

No. The linearity property can only be applied to constant coefficients, not time-dependent ones. – 2 points

c. If $x(t) = e^{t-2}u(t-2)$, which of the following is the Laplace transform of $x(t)$?

(a) $\frac{1}{s-2}e^{-s}$, (b) $\frac{1}{s+1}e^{-2s}$, (c) $\frac{1}{s-1}e^{-2s}$, (d) none of the above.

We can break it down into two steps:

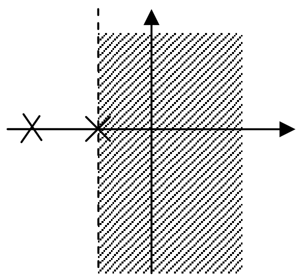
e^{t-2} u(t-2) ← time delay ← e^{t} u(t) ← multiplication of exponential ← u(t)

Since L[u(t)] = 1/s, the complex shifting theorem tells us that L[e^{t}u(t)] = 1/(s-1). Finally, time delay corresponds to multiplication of exponential in s-domain, i.e. L[e^{t-2}u(t-2)] = e^{-2s}/(s-1) or (c).

8. Laplace Transform

a. $x(t)$'s Laplace transform is $X(s) = \frac{s+4}{s^2+3s+2}$. Draw its ROC.

After simplification, X(s) = 1/[(s+1)(s+2)]. The poles of X(s) are -1 and -2. (2 pts)
The ROC looks like



b. Find the differential equation relating the input x(t) and the output y(t) if the transfer function of the system is given as follows:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+2s+1}$$

$$H(s) = \frac{1}{s^2+2s+1} \Rightarrow s^2Y(s) + 2sY(s) + Y(s) = X(s)$$

By the definition of transfer function, all initial conditions are assumed to be zero. Taking inverse Laplace transform, we get

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = x(t)$$