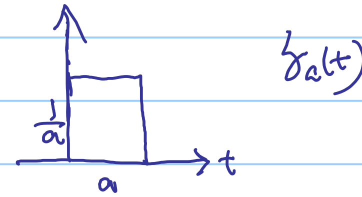
Lecture 15

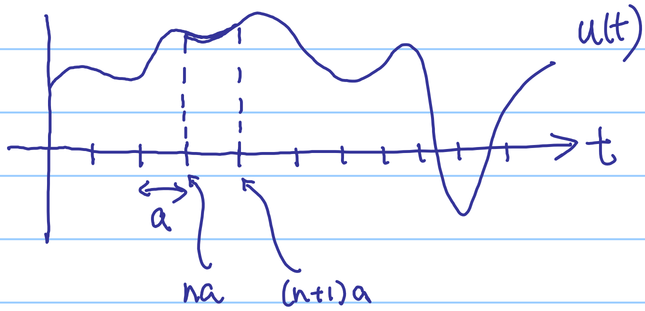
Note Title 10/1/2007

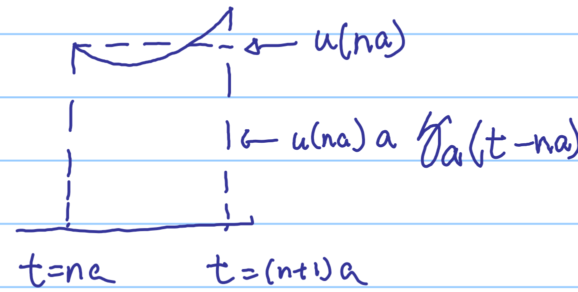
- Continuous-time convolution
- Differential Equation — ODE is LTI system
- Orthonormal basis, Eigen basis

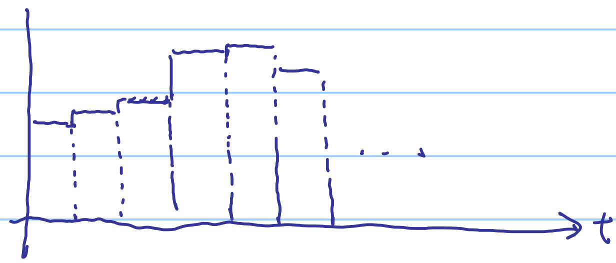


Continuous-Time Convolution

Assume $\mathcal{L}\{u(t)\} = y(t)$ is a CT LTI system







piece-wise constant approximation of $u(t)$:

$$\tilde{u}_a(t) = \sum_{n=-\infty}^{\infty} u(na)\, a\, \gamma_a(t-na)$$

$$* \quad u(t) = \lim_{a \to 0} \tilde{u}_a(t)$$

$$\Rightarrow \quad \mathcal{L}[u(t)] = \lim_{a \to 0} \mathcal{L}[\tilde{u}_a(t)]$$

$$\mathcal{L}[\tilde{u}_a(t)] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} u(na)\, a\, \gamma_a(t-na)\right]$$

$$= \sum_{n=-\infty}^{\infty} \mathcal{L}[u(na)\, a\, \gamma_a(t-na)]$$

$$= \sum_{n=-\infty}^{\infty} u(na)\, \mathcal{L}[\delta_a(t-na)]\, a$$

integration variable

$a \to 0 \Rightarrow na \to \tau$, $a \to d\tau$, $\Sigma \to \int$

$$\mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(\tau)\, \mathcal{L}[\delta(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

where $h(t) = \mathcal{L}[\delta(t)]$ is the impulse response

Java template

$$\boxed{\int_{-\infty}^{\infty} h(v)\, x(t-v)\, dv}$$

— This is the CT time convolution formula.

Make sure you know how to do CT convolution graphically!

Examples of CT LTI systems

Most of the physical systems can be described by an Ordinary Differential Equation:

ex
$$3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = 3u(t) \qquad (\Delta)$$

— ordinary = all the coefficients are constants — they are not time dependent.

$\Rightarrow$ ex, LRC, op-amp, Newtonian physics

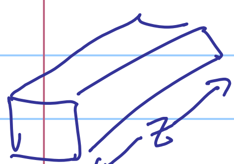
ODE describe a lumped system

$$y(t) = -3\frac{dy}{dt^2} - 2\frac{dy}{dt} + 3u(t)$$

finite states !!

Example of a non-lumped system (distributed system):

- non-causal CT system is not lumped.
- Heat transfer or fluid dynamics



$$y(t,z) = 3\frac{\partial y}{\partial t} + 2\frac{\partial y}{\partial z} + u(t,z)$$

Not finite $\frac{\partial y}{\partial t}(z)$

ODE describes a LTI system

① Linearity

$$\begin{cases} 3\frac{d^2y_1}{dt} + 2\frac{dy_1}{dt} + y_1(t) = 3u_1(t) & (1) \\ 3\frac{dy_2}{dt} + 2\frac{dy_2}{dt} + y_2(t) = 3u_2(t) & (2) \end{cases}$$

$au_1(t) + bu_2(t) \rightarrow \boxed{\text{Sys}} \rightarrow$??

$$a\cdot(1) + b\cdot(2) \Rightarrow 3\frac{d^2}{dt^2}\left(ay_1(t) + by_2(t)\right) + 2\frac{d}{dt}\left(ay_1(t) + by_2(t)\right)$$

$$+ \left(ay_1(t) + by_2(t)\right) = 3\left(au_1(t) + bu_2(t)\right)$$

$\Rightarrow$ Output $= ay_1(t) + by_2(t)$ ✓

② Time Invariance

$t \to t+T$ $\quad 3\frac{d^2y}{dt^2}\Big|_{t=(t+T)} + 2\frac{dy}{dt}\Big|_{t=(t+T)} + y(t+T) = 3u(t+T)$

$$\Rightarrow \mathcal{L}(u(t+T)) = y(t+T)$$

ex $\quad t^2 \frac{dy}{dt} = 3u(t)$

$t \to t+T$ $\quad (t+T)^2 \frac{dy}{dt}\Big|_{t=t+T} = 3u(t+T)$

cofficient changes $\Rightarrow$ different solution $\Rightarrow$ NOT TI

$\Rightarrow$ ODE is TI because all cofficients are constant !!

Δ

ODE is a LTI system $\Rightarrow$ the $y_{zs}(t)$ of ODE can be solved by time convolution

Representation of a signal as a weighted sum of sinusoids.

① Natural – true but there are other representations that are more natural in specific problems

② Complete orthonormal basis representation.
(Invertible $\Leftrightarrow$ not losing any information)

③ Sinusoids are EIGEN functions of LTI systems

$$A\sin(wt+\theta) \longrightarrow \boxed{LTI} \longrightarrow \tilde{A}\sin(wt+\tilde{\Phi})$$

Exactly the same frequency back

$$Y(jw) = H(jw) \cdot X(jw)$$