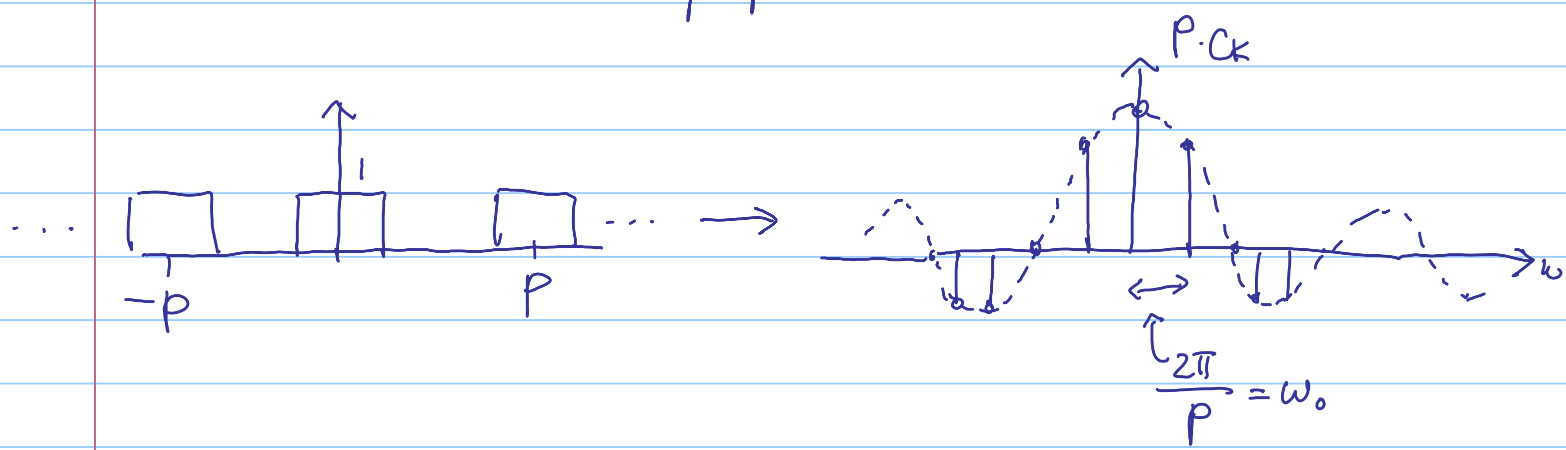
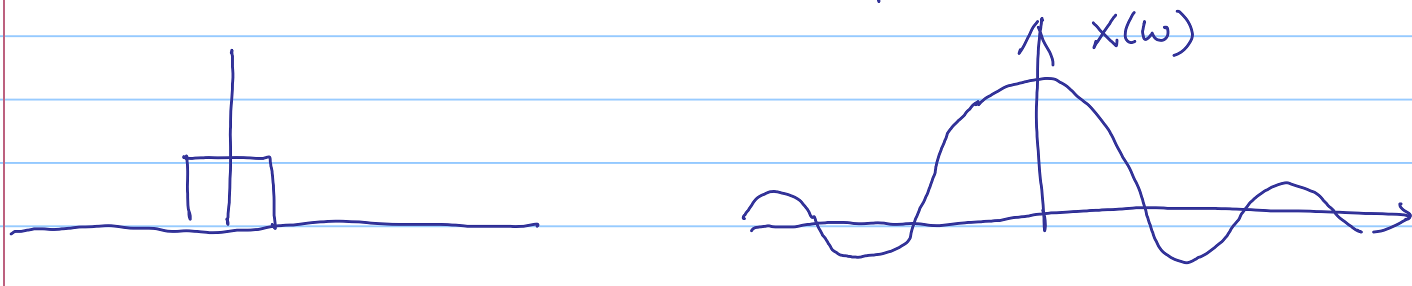
Lecture 20

Note Title 10/17/2007

- Fourier Transform — formula
  - examples
  - properties



$P \to \infty$ $\qquad\qquad$ $\frac{2\pi}{P} \to 0$



Fourier Transform

$$X(\omega) \triangleq P\, C_k \qquad\qquad P \to \infty$$

$$= \cancel{P} \cdot \frac{1}{\cancel{P}} \int_{\langle P \rangle} x(t)\, e^{-jn\omega_0 t}\, dt \qquad\qquad P \to \infty$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad\qquad n\omega_0 \triangleq \omega$$

— Analysis Formula $\quad$ ($x(t)$ aperiodic)

Inverse Fourier Transform $(X(\omega) \rightarrow x(t))$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \text{For periodic } x(t)$$

$$= \frac{1}{P} \sum_{k=-\infty}^{\infty} P c_k e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \omega_0 \sum_{k=-\infty}^{\infty} X(\omega) e^{jk\omega_0 t}$$

As $P \rightarrow \infty$ $\quad \omega_0 \rightarrow d\omega \quad k\omega_0 \rightarrow \omega \quad \Sigma \rightarrow \int$

Aperiodic $\quad x(t) = \frac{1}{2\pi} d\omega \int X(\omega) e^{j\omega t} = \frac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega$

- Synthesis Formula

Space: aperiodic $x(t)$

Orthonormal basis: $\frac{1}{\sqrt{2\pi}} e^{j\omega t}$ for all possible ω.

- uncountably many basis functions

vs FS: $e^{j0t}, e^{\pm j\omega_0 t}, e^{\pm j2\omega_0 t}, \ldots$ Countably Infinite

$x(t), y(t) \longrightarrow \tilde{X}(\omega) \quad \tilde{Y}(\omega)$

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t)\, dt \qquad \tilde{X}(\omega) = \langle x(t), \tfrac{1}{\sqrt{2\pi}} e^{j\omega t} \rangle$$

sum with integral
∵ uncountably many basis

$$x(t) = \int \underbrace{\tilde{X}(\omega)}_{\text{coefficient}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{j\omega t}}_{\text{orthonormal basis.}} d\omega$$

factor

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} X(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int \tilde{X}(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega$$

$$\langle x(t), y(t) \rangle = \langle \tilde{X}(\omega), \tilde{Y}(\omega) \rangle$$

$$\Rightarrow \text{Energy}(x(t)) = \langle x(t), x(t) \rangle = \langle \tilde{X}(\omega), \tilde{X}(\omega) \rangle$$

$$\text{LHS} = \langle x(t), x(t) \rangle = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$\text{RHS} = \langle \tilde{X}(\omega), \tilde{X}(\omega) \rangle$$

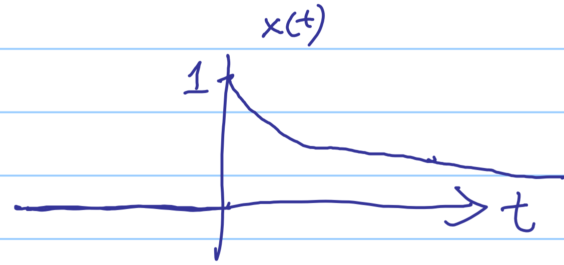
$$= \int_{-\infty}^{\infty} \tilde{X}(\omega)\, \tilde{X}^*(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} X(\omega)\right)\left(\frac{1}{\sqrt{2\pi}} X^*(\omega)\right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \text{LHS} = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Parseval's relationship

Example: $x(t) = e^{-t} u(t)$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-t} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-t-j\omega t} dt$$

$$= \frac{1}{-1-j\omega}\left[e^{-t-j\omega t}\right]_0^{\infty} = \frac{1}{-1-j\omega}\left[0 - 1\right]$$

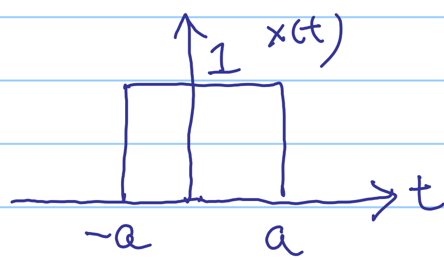
$$= \frac{1}{1+j\omega}$$

$$\lim_{t\to\infty} e^{-t-j\omega t}$$

$$= \lim_{t\to\infty} e^{-t} e^{-j\omega t}$$

$$= \lim_{t\to\infty} e^{-t}\left(\cos(\omega t) - j\sin(\omega t)\right) = 0$$

ex/



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-a}^{a} e^{-j\omega t} dt$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-a}^{a}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega a} - e^{j\omega a}\right]$$

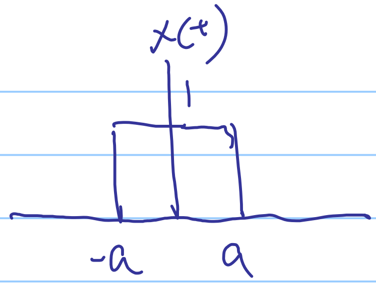
$$= \frac{2\sin(\omega a)}{\omega} \leftarrow \text{Real}$$

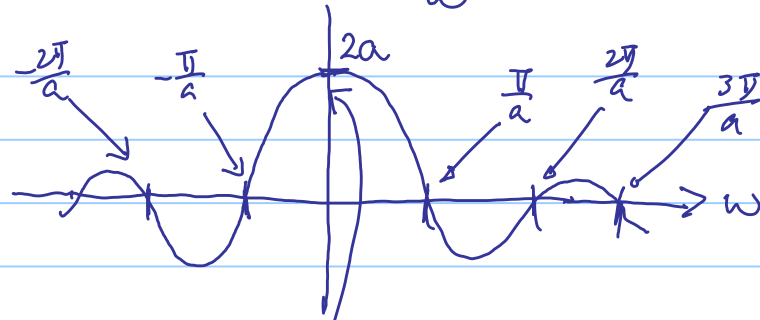
$$\text{sinc}(\omega) = \frac{\sin \omega}{\omega}$$

$$\frac{2\sin(\omega a)}{\omega} = 2\,\text{sinc}(\omega a)\cdot a$$

$$= 2a\,\text{sinc}(\omega a)$$

$$X(\omega) = \frac{2\sin(\omega a)}{\omega}$$

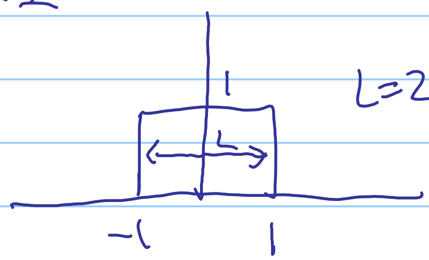




$$X(\omega=0) = \lim_{\omega \to 0} \frac{2\sin(\omega a)}{\omega} \begin{matrix} \leftarrow \frac{d}{d\omega} \\ \leftarrow \frac{d}{d\omega} \end{matrix}$$

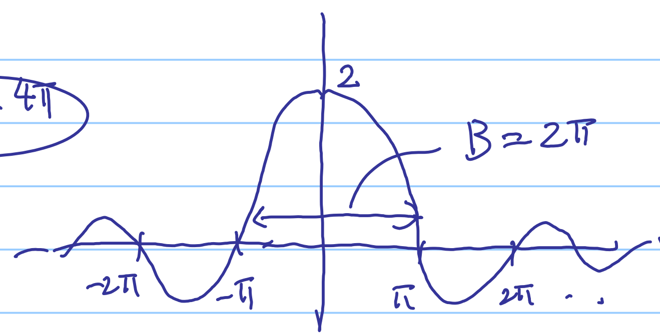
$$= \lim_{\omega \to 0} \frac{2a\cos(\omega a)}{1} = 2a$$

$a = 1$

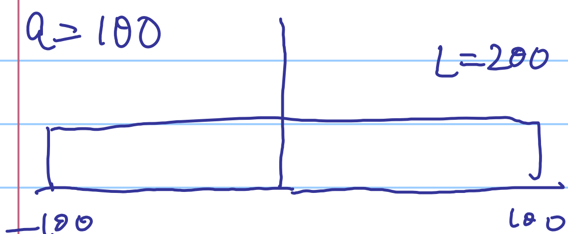


$L = 2$

$B \cdot L = 4\pi$

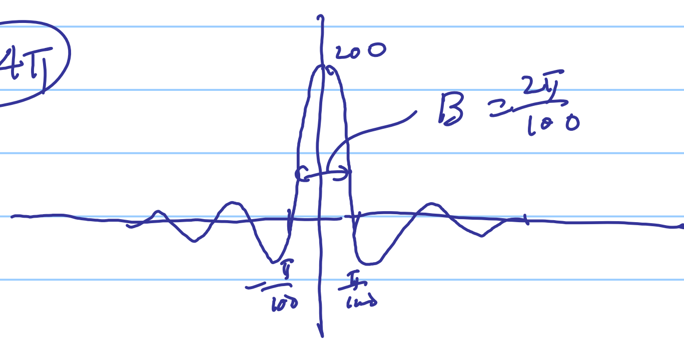


$a = 100$

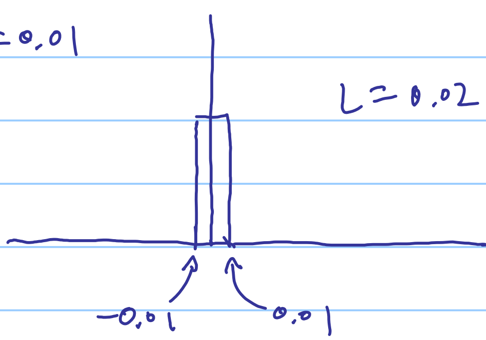


$L = 200$

$B \cdot L = 4\pi$

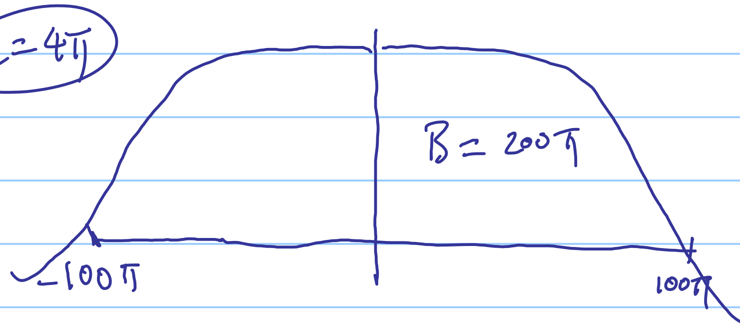


$a = 0.01$



$L = 0.02$

$B \cdot L = 4\pi$



L = duration of the signal (ex 1-D position of a particle)

B = "width" of the spectrum

In general $\quad B \cdot L \geq k\pi \quad$ (k constant)

FT of a position = momentum. $\left(\frac{d}{dt}\right)$

$\Updownarrow$

Heisenberg Uncertainty principle.

frequency scaling :

$$x(t) \longrightarrow X(\omega)$$

$$x(at) \longrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

scale in exactly the opposite way

compress ($a > 1$) $\longleftrightarrow$ expanding ($\frac{1}{a} < 1$)