Lecture 22

- short HW 8 due Friday (posted later today)
- Midterm 2 next Wednesday
- from last midterm to the end of Ch.4

⑤ Convolution in time = Multiplication in freq
Dual: Convolution in freq = Multiplication in time

\[ u(t) \rightarrow \begin{bmatrix} CT \linebreak LTI \linebreak h(t) \end{bmatrix} \rightarrow y_{2s}(t) = \int u(\tau) h(t-\tau) \, d\tau \]

\[ \Downarrow \text{F} \]

\[ u(\omega) \rightarrow \begin{bmatrix} \overline{H(\omega)} \end{bmatrix} \rightarrow y_{2s}(t) = \mathcal{F}^{-1} [U(\omega)H(\omega)] \]

This is important

① Convolution is important

② To calculate \( y(t) \) at a particular time \( t \):
   ① Multiply \( u(\tau) h(t-\tau) \) for \( \tau \)
   ② Integration \( \int u(\tau) h(t-\tau) \, d\tau \)
It's much easier to do it in $\omega$-domain, i.e., for each $\omega$:

$$Y(\omega) = U(\omega) H(\omega)$$

For those who have taken MA 322:

$$V_\omega = e^{j\omega t}$$

**Linear Transform** $L[u] = v$ if $u, v$ are 2-D vectors, $L[.]$ is a multiplication of a $2 \times 2$ matrix.

**Special Vector**:

$$L[v] = \lambda v$$

$\lambda$ - eigenvalue

$v$ - eigenvector

$L = 2 \times 2$ matrix, we have two independent eigenvectors $V_1, V_2$ with eigenvalues $\lambda_1, \lambda_2$

For any vector $u$, $u = aV_1 + bV_2$
\[ L[u] = L[a v_1 + b v_2] \]
\[ = a L[v_1] + b L[v_2] \]
\[ = a \lambda_1 v_1 + b \lambda_2 v_2 \]

In the direction of \( v_1 \) \( a \xrightarrow{\text{L}} a \lambda_1 \) (scalar multiplication)

In the direction of \( v_2 \) \( b \xrightarrow{\text{L}} b \lambda_2 \) (No need to do any matrix multiplication)

\[
\begin{array}{c}
\text{u(t)} \\
\xrightarrow{\text{LTI}} \\
\xrightarrow{\text{y(t)}}
\end{array}
\]

\[ V_w(t) = e^{j\omega t} \text{ is an eigenvector/eigenfunction for this LTI system for each } w. \]

\[ L[V_w(t)] = L[e^{j\omega t}] = \lambda_w e^{j\omega t} \]

\[ \lambda_w \text{ is eigenvalue for } V_w(t) = H(w) \]

\[ = \text{DT of impulse response} \]

\[ \text{uncontably many eigenvectors.} \]
Given an input \( u(t) \) to \( L[\cdot] \),

\[
y(t) = L[u(t)] \quad \text{Invertible FT}
\]

\[
y(t) = L\left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} U(w) e^{jwt} \, dw \right]
\]

Linear property (generalized summation)

\[
L_{\text{eigenvec}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(w) L[e^{jwt}] \, dw.
\]

The eigen-vector of \( L \)

\[
L = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(w) H(w) e^{jwt} \, dw
\]

\[
= \mathcal{F}^{-1}[U(w) H(w)]
\]

\[\text{ex/} \]

\[
\begin{array}{c}
\begin{array}{c}
\text{u(t)} \quad \text{R} \\
\text{H} \quad \text{C} \\
\text{y(t)}
\end{array}
\end{array}
\]

\[
U(w)
\]

\[\text{w} \]

\[\text{0} \quad \pi \]

\[\text{R} \quad \text{C} \quad \text{y(t)} \]

\[\text{u(t)} \quad \text{H} \quad \text{U(w)} \]

\[\text{0} \quad \pi \quad \text{w} \]
Can the output spectrum \( Y(w) \) look like the following?

\[
Y(w)
\begin{array}{c}
-\infty \leftarrow \\
\rightarrow \infty
\end{array}
\]

According to multiplication in freq = conv in time:

\[
Y(w) = U(w) \cdot H(w)
\]

\[
\begin{array}{c}
Y(w)
\end{array}
\begin{array}{c}
\leftarrow \\
\rightarrow \infty
\end{array}
\overset{?}{=}
\begin{array}{c}
\frac{1}{T}
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
-\pi
\end{array}
\]

\[
\times \text{ Not possible } \quad Y(w) = 0 \quad w > \frac{\pi}{T}
\]

\[
\times \text{ Not possible } \quad Y(w) = 0 \quad w < -\frac{\pi}{T}
\]

\[
\Rightarrow \text{ a LTI system CANNOT CREATE NEW FREQUENCY SPECTRUM} \]

!!
Dual: \[ \mathcal{F} \left[ U_1(t) U_2(t) \right] = \int_{-\infty}^{\infty} U_1(\omega) U_2(\omega - \omega_0) d\omega. \]

Multiplication in time \(\Rightarrow\) convolution in freq

Ex/ Design a low-pass filter.

Ideal low-pass

\[ \mathcal{F}^{-1} \]

\[ \text{sinc} \]

\(\Leftarrow\) Anticausal

\(\Rightarrow\) Not implementable

In real-life, we have to truncate \( h(t) \)
In frequency, convolution in freq

Spectrum after the truncation

= 

high freq ripples

↓

high freq noise to pass through

overshoot near singularities "Gibbs phenomenon"