Today

- Discrete Fourier Transform
- Motivate why we need more transform techniques (Ch6, Ch8)
- Laplace transform
- Region of Convergence for LT
- LT in used to model LTI system

Discrete Fourier Transform (DFT)

- not to be confused with DTFI

\[ X_k = X_d \left( \frac{2\pi}{TN} k \right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \left( \frac{2\pi}{TN} k \right) n T} \]

\[ = \sum_{n=-\infty}^{\infty} x[n] e^{-j \frac{2\pi}{N} k n} \]

\# of freq sample points

\[ \leftarrow \text{Time index} \]

\[ \uparrow \text{Freq index} \]

\[ \text{freq sample points} \]
If \( x[n] \) has equal or fewer than \( N \) points, the above equation is invertible

Analysis: \( X_k = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} \)

Synthesis: \( x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi nk}{N}} \)

From a vector space point-of-view:

\[
B_k[n] = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi nk}{N}} \quad k = 0, 1, 2, ..., N-1
\]

forms an orthonormal basis set

Advantages of DFT

1. Both time-domain and frequency-domain rep of DFT are discrete \( \Rightarrow \) good for computer

2. DFT has a very efficient implementation called Fast Fourier Transform

Direct computation, you need \( \sim N^2 \) operations

complex addition & multiplication
Fast Fourier Transform, you need ~ N \cdot \log N \text{ operations}

⇒ High computational savings!

Why need more transforms?

1. \[ \text{Input} \rightarrow [\text{LTI}] \rightarrow \text{Output} \]

\[ y(t) = y_{zs}(t) + y_{zi}(t) \]

\[ \uparrow \quad \text{Zero state} \quad \uparrow \text{Zero input} \]

⇒ Output due to input \Rightarrow Output due to initial state

\[ y_{zs}(t) = \int u(\tau) h(t-\tau) \, d\tau \]

Laplace (in \( s \)) and Z-transform (in \( \mathcal{Z} \)) allow us to study \( y_{zi}(t) \) also.
(2) General system properties — Stability

BIBO stability = Bounded input, Bounded output.

\[
\text{Bound input} \xrightarrow{\text{LTI}} \text{Bound output} \quad \text{constant}
\]

\[x(t) = |x(t)| < M \quad \text{for the entire signal.}\]

\[\begin{align*}
\text{FI} \\
x(t) & = e^t g(t) \\
\checkmark & = e^t g(t) \\
\checkmark & = \delta(t) \\
\checkmark & = \sin(t) \\
\checkmark & = t g(t)
\end{align*}\]

Laplace, Z-transform allows us to better study the stability of system.
Laplace

\[ x(t) = e^t g(t) \quad \text{FT does not exist !!} \]

\[ x(t) \rightarrow \mathcal{X} \rightarrow x(t) e^{-\delta t} = x_\delta(t) \]

Damping signal \( (e^{-\delta t}) \) \( \delta \) is real

- this operation is reversible :: you can always divide \( x_\delta(t) \) by \( e^{-\delta t} \) to get back \( x(t) \)

If \( \delta = 1 \)

\[ x_\delta(t) = x(t) \cdot e^{-t} \]

\[ = e^t \cdot g(t) \cdot e^{-t} \]

\[ = g(t) \rightarrow \text{FT exists}! \]

In fact that if \( \delta \geq 1 \), the FT of \( x_\delta(t) \) always exists!

\[ \mathcal{F}[x_\delta(t)] = \int_{-\infty}^{\infty} x_\delta(t) e^{-j\omega t} dt \]
\[ = \int_{-\infty}^{\infty} x(t) \ e^{-st} \cdot e^{j \omega t} \ dt \]
\[ = \int_{-\infty}^{\infty} x(t) \ e^{-(s+j\omega) t} \ dt \]
\[ = \int_{-\infty}^{\infty} x(t) \ e^{-st} \ dt \ = \mathcal{L}[x(t)] \]