Lecture 29

Note Title 11/9/2007

No lecture Next Wed (video-taped makeup next Thurs during discussion)

Today
- Laplace Transform
- ROC
- Examples
- LT of ODE $\Rightarrow$ Transfer functions

Laplace Transform is the Fourier Transform of the "damped" version of a signal

controlled by σ

$$\mathcal{F}[e^{-\sigma t}\cdot x(t)] = \int_0^{\infty} e^{-\sigma t} x(t)\, e^{-j\omega t}\, dt$$

unilateral Laplace Transform

Large the σ, more damping

$\Downarrow$

$\mathcal{F}$ will continue to exist

$$= \int_0^{\infty} x(t)\, e^{-\underbrace{(\sigma+j\omega)}_{s} t}\, dt$$

$$= \int_0^{\infty} x(t)\, e^{-st}\, dt \triangleq X(s)$$

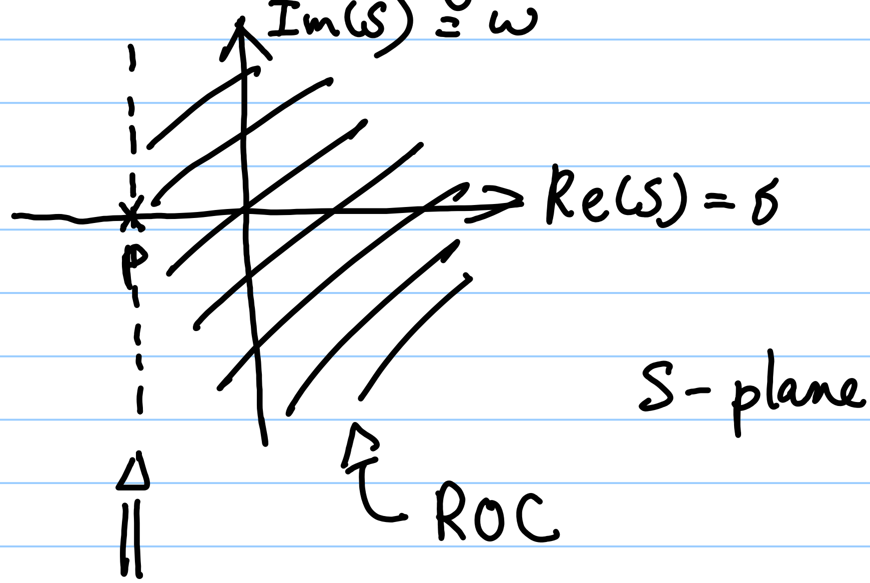
Laplace Transform
- ① Evaluation of the integral
- ② Specify the range of σ such the the Fourier Transform exists

② $\Rightarrow$ Region of Convergence (in s)

$\sigma = Re(s)$

some value.

ROC is in the form of $s: Re(s) = \sigma > p$



ROC does not contain the boundary.

Mathematically, ROC contains all complex value s such that the following holds

$$\int_0^{\infty} |x(t) e^{-st}| \, dt < \infty$$

ex/ $x(t) = e^{t} q(t)$ Find $X(s)$

$$X(s) = \int_0^{\infty} x(t) e^{-st} \, dt$$

$$= \int_0^\infty e^t e^{-st}\, dt$$

$$= \int_0^\infty e^{(1-s)t}\, dt$$

$$= \frac{1}{1-s}\left[e^{(1-s)t}\right]_0^\infty$$

$$= \frac{1}{1-s}\left[\lim_{t\to\infty} e^{(1-s)t} - 1\right]$$

$$\lim_{t\to\infty} e^{(1-\sigma-j\omega)t} \qquad s = \sigma + j\omega$$

oscillating

$$= \lim_{t\to\infty} e^{(1-\sigma)t} \cdot e^{-j\omega t}$$

Three cases

Case 1: $1-\sigma > 0 \Rightarrow e^{(1-\sigma)t} \nearrow$ as $t\to\infty$

$$\Rightarrow \lim_{t\to\infty} e^{(1-\sigma)t} \cdot e^{-j\omega t} = \infty \quad \times$$

$\Rightarrow$ Case 2: $1-\sigma < 0 \Rightarrow e^{(1-\sigma)t} \searrow$ as $t\to\infty$

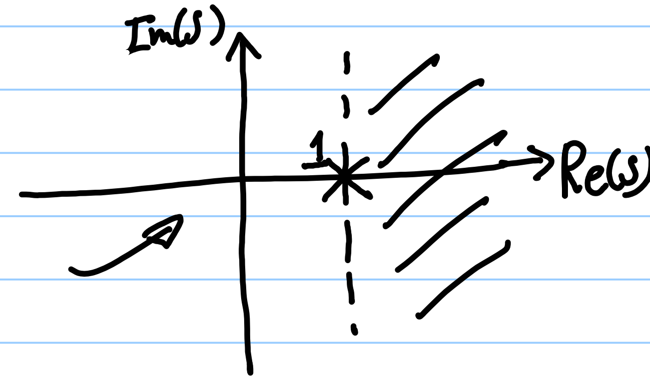
$$\Rightarrow \lim_{t\to\infty} e^{(1-\sigma)t} \cdot e^{-j\omega t} = 0 \quad \checkmark$$

Case 3 : $1-\sigma=0 \Rightarrow e^{(1-\sigma)t}=1$

$\Rightarrow \lim\limits_{t\to\infty} e^{-j\omega t}$ = don't know

$X(s)=\dfrac{1}{s-1}$ if $\sigma = Re(s) > \underline{1}$

ROC :



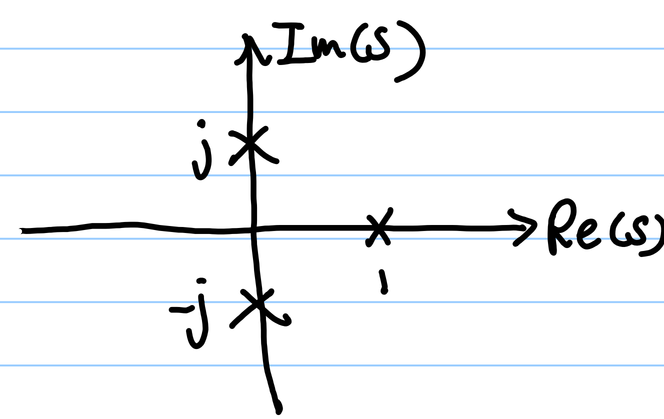
$X(s)\Big|_{s=1}=\infty$

$\Rightarrow$ we call $s=1$, a pole of $X(s)$

Fact 1: The boundary of the ROC must fall on a pole

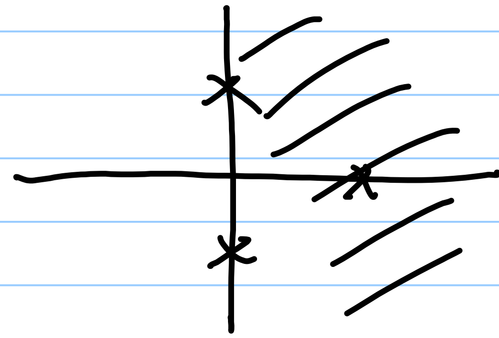
$$X(s)=\frac{1}{(s-1)(s^2+1)}=\frac{1}{(s-1)(s+j)(s-j)}$$

Poles: $1,\ +j,\ -j$

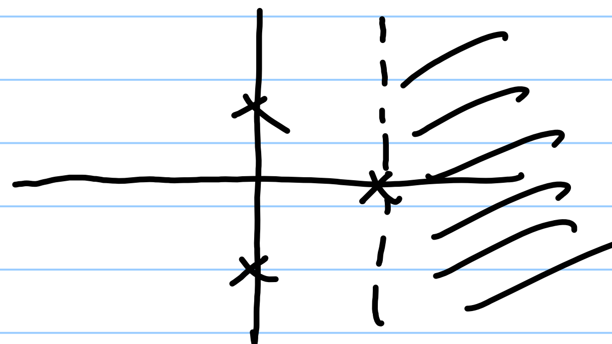


What does its ROC look like

Choice A :



Choice B :



Since ROC cannot contains any s where $X(s) = \infty$, the answer is choice B

Fact 2 : The boundary of the ROC must fall on the <u>RIGHTMOST</u> pole(s)

ex/ Find $X(s)$ for $x(t) = u(t)$

$$X(s) = \int_0^{\infty} 1 \cdot e^{-st}\, dt = \frac{1}{-s}\left[e^{-st}\right]_0^{\infty}$$

$$= \frac{1}{-s}\left[0 - 1\right] = \frac{1}{s}$$

$$\text{ROC} : \{s : \text{Re}(s) > \underset{\text{pole}}{0}\}$$

ex Find $X(s)$ for $x(t) = \delta(t)$

$$X(s) = \int_{0}^{\infty} \delta(t)\, e^{-st}\, dt$$

By convention, we include the origin [denote the lower limit as 0^-]

$$= e^{-s \cdot 0} = 1$$

No pole : ROC is the entire s-plane

<u>Using Laplace Transform to study lumped LTI system</u>

$\Rightarrow$ ODE

ex/ System is described by the follow ODE

$$u(t) \longrightarrow \boxed{\mathcal{L}} \longrightarrow y(t)$$

$$\frac{dy}{dt} + 2y(t) = 4u(t)$$

Laplace Transform

$$\mathcal{L}\left[\frac{dy}{dt} + 2y(t)\right] = \mathcal{L}\left[4u(t)\right]$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 2Y(S) = 4U(s)$$

$\Uparrow$

??

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty \frac{dy}{dt} e^{-st}\, dt$$

Integration by parts

$\int u\,dv = uv - \int v\,du$

$$= \int_0^\infty e^{-st}\, dy$$

$$= \left[e^{-st} y\right]_0^\infty - \int_0^\infty y\, de^{-st}$$

$$= \left[0 - y(0^-)\right] - \int_0^\infty y(t) \frac{1}{-s} e^{-st}\, dt$$

$$= -y(0^-) + \frac{1}{s} \underbrace{\int_0^\infty y(t) e^{-st}\, dt}_{Y(s)}$$

$$= \frac{1}{s} Y(s) - y(0^-)$$