Lecture 31

Note Title 11/12/2007

Today
- Laplace transform of derivative
- LT on generic ODE
  ⇒ separate y_{ZS} and y_{ZI}
- LT on circuit elements (complex impedances)
- Inverse LT transform

Last time:

Given $\mathcal{L}[x(t)] = X(s)$

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = ?$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

$$= \int_{0^-}^{\infty} \underline{\frac{d}{dt}x(t)} \cdot e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} e^{-st}\, dx \qquad \frac{d}{dt}e^{-st} = \frac{de^{-st}}{d{-st}} \cdot \frac{d{-st}}{dt}$$

$$= \left[x(t) \cdot e^{-st}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} x(t)\, de^{-st} \qquad e^{-st} \cdot (-s)$$

$$= \lim_{t\to\infty} x(t) \cdot \underbrace{e^{-st}}_{0\ (s \in ROC)} - x(0^-) \cdot \underbrace{e^{-s \cdot 0^-}}_{1} - \int_{0^-}^{\infty} x(t)(-s) \cdot e^{-st}\, dt$$

$$= -x(0^-) + s\int_{0^-}^{\infty} x(t) e^{-st}\, dt$$

$$= s X(s) - x(0^-)$$

LT of $x(t)$ ⟶ $X(s)$

$x(0^-)$ ⟵ $x(t)\big|_{t=0^-}$ constant = Initial Condition we control.

ex/ $$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{du}{dt} + u(t) \qquad \overset{\downarrow}{u(t)} \to \boxed{LTI} \to y$$

Need initial condition

① $u(0^-) = 0$ ← we also assume IC of the input as zero

② $y(0^-)$, $\left.\frac{dy}{dt}\right|_{t=0^-}$ ← Two IC to fully describe the internal state of the LTI system

Step 1 : LT on the differential Equation

$$\mathcal{L}\left[\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t)\right] = \mathcal{L}\left[\frac{du}{dt} + u(t)\right]$$

$$s\mathcal{L}\left[\frac{dy}{dt}\right] - y'(0^-) + 3\left[sY(s) - y(0^-)\right] + 2Y(s) = sU(s) - \cancel{u(0^-)}^{\,0} + U(s)$$

$$s\left[sY(s) - y(0^-)\right] - y'(0^-) + 3sY(s) - 3y(0^-) + 2Y(s) = sU(s) + U(s)$$

$$Y(s)\left[s^2 + 3s + 2\right] - y(0^-)\left[s+3\right] - y'(0^-) = U(s)(s+1)$$

$$Y(s) = U(s)\frac{s+1}{s^2+3s+2} + \frac{y(0^-)(s+3) + y'(0^-)}{s^2+3s+2}$$

① If the initial state $y(0^-) = 0$ and $y'(0^-) = 0$ (Zero State)

$$Y(s) = U(s)\frac{s+1}{s^2+3s+2}$$ — LT of the zero state response of the system

"Transfer" Function

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{du}{dt} + u(t)$$

input

Call $H(s) \triangleq \dfrac{s+1}{s^2+3s+2}$ output part .... Rational Function of s $\triangleq$ ratio of 2 polynomials of s

$$Y(s) = U(s) \cdot H(s)$$

Recall In FT, $Y(\omega) = U(\omega) \cdot H(\omega)$

$$H(\omega) = \mathcal{F}[\text{Impulse Response } h(t)]$$

$$H(s) = \mathcal{L}[\text{Impulse Response } h(t)]$$

Corollary : Convolution in time $\Longleftrightarrow$ Multiplication in Laplace domain

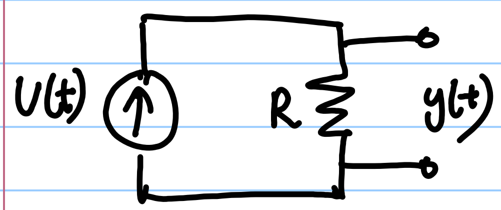
$$y_{zs}(t) = \int u(\tau)h(t-\tau)d\tau \qquad Y_{zs}(s) = U(s)H(s)$$

Corollary : e^{-st} (s is a complex number) is an eigensignal of any LTI system
(Replace $j\omega$ with $s = \sigma + j\omega$, and the proof is exactly the same)

② Zero-input response : $U(s) = 0$

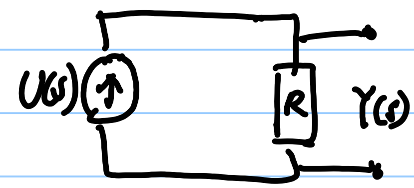
$$Y(s) = \frac{y(0^-)(s+3) + y'(0^-)}{s^2+3s+2} = \mathcal{L}[y_{ZI}(t)]$$

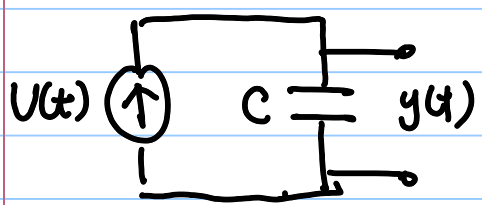
- denominator is the same as $H(s)$
$\Rightarrow$ poles = roots of the denominator
$\Rightarrow$ ROC of $Y(s)$ ($H(s)$)



$$y(t) = U(t)R$$

$$Y(s) = U(s)R$$





$$i_c(t) = C\frac{dv_c}{dt}$$

$$U(t) = C\frac{dy}{dt}$$

$$U(s) = CsY(s) - Cy(0^-)$$

$$Y(s) = \frac{1}{Cs}U(s) + \frac{1}{s}y(0^-)$$