Lecture 31

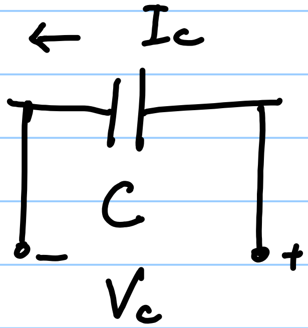
Note Title 11/15/2007

	Properties	Time Domain	Laplace Transform
1	Linearity	$a_1x_1(t)+a_2x_2(t)+\ldots+a_nx_n(t)$	$a_1X_1(s)+a_2X_2(s)+\ldots+a_nX_n(s)$
2	Frequency Shifting	$e^{-\alpha t}x(t)$	$F(s+\alpha)$
3	Time Delay	$x(t-a)u(t-a)$	$e^{-as}X(s)$
4	Time Scaling	$x(\alpha t)$	$\frac{1}{\alpha}X\left(\frac{s}{\alpha}\right)$
5	Time Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
6	Time Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{X(s)}{s}+\frac{1}{s}\int_{-\infty}^{0^-}x(\tau)d\tau$
7	~~Initial Value Theorem~~	$\cancel{\lim_{t\to 0^+}x(t)}$	$\cancel{\lim_{s\to\infty}sX(s)=x(0^+)}$ ×
8	~~Final Value Theorem~~	$\cancel{\lim_{t\to\infty}x(t)}$	$\cancel{\lim_{s\to 0}sX(s)=x(\infty)}$ ×
9	Time Convolution	$x(t)*y(t)$	$X(s)Y(s)$

Signal	Laplace Transform
1. $\delta^{(n)}(t)$	s^n
2. 1 or $u(t)$	$\frac{1}{s}$
3. $\frac{t^n\exp(-\alpha t)u(t)}{n!}$	$\frac{1}{(s+\alpha)^{n+1}}$
4. $\cos\omega_0 t\, u(t)$	$\frac{s}{s^2+\omega_0^2}$
5. $\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
6. $\exp(-\alpha t)\cos\omega_0 t\, u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$
7. $\exp(-\alpha t)\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$

Today

- Laplace models for capacitor & inductor
- example
- properties of LT
- Inverse Laplace Transform



$$i_c(t) = C\frac{dV_c(t)}{dt}$$

$$\Big\downarrow \mathcal{L}$$

$$\underline{I_c(s)} = C\left[sV_c(s) - v_c(0^-)\right]$$

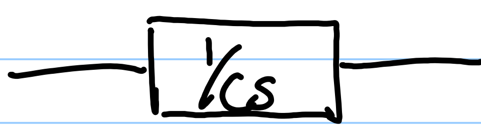
$$= Cs\,\underline{V_c(s)} - CV_c(0^-)$$

If $V_c(0^-) = 0$ (zero IC)

$$I_c(s) = Cs\,V_c(s)$$

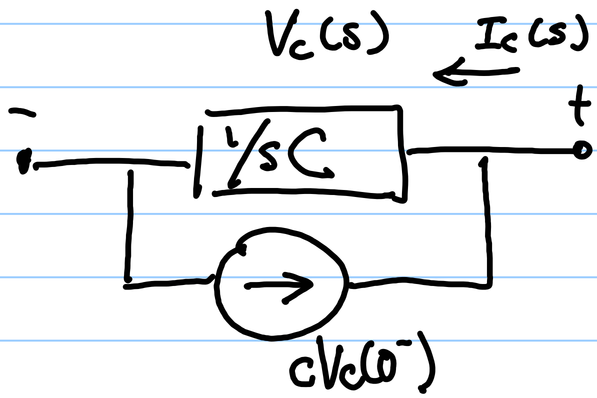
$$\Rightarrow \frac{V_c(s)}{I_c(s)} = \frac{1}{Cs} \longleftarrow$$ looks like a "resistor"

Complex Impedance



If $V_c(0^-) \neq 0$, treat it as a current source

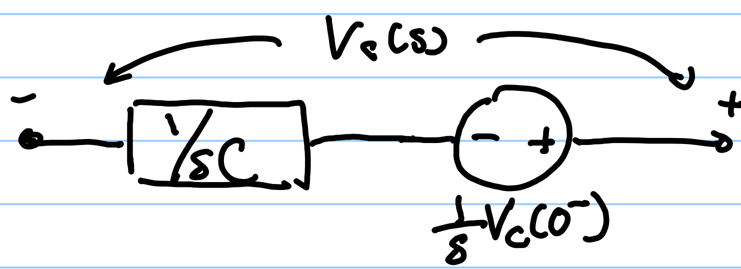
$$I_c(s) = sC\,V_c(s) - C V_c(0^-)$$



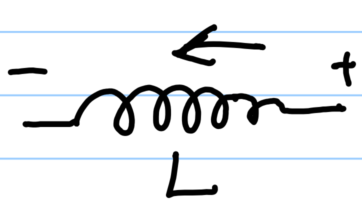
Norton equivalent

voltage source

$$V_c(s) = \frac{1}{sC} I_c(s) + \frac{1}{s} V_c(0^-)$$



Inductor



$$v_L(t) = L\frac{d\,i_L(t)}{dt}$$

$\mathcal{L}\{$

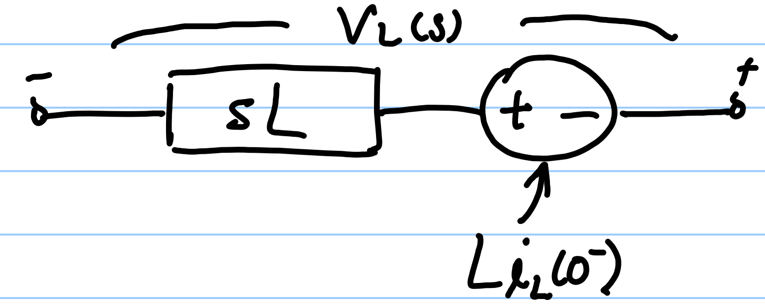
$$V_L(s) = L\left[sI_L(s) - i_L(0^-)\right]$$

$$= sL\,I_L(s) - L\,i_L(0^-)$$

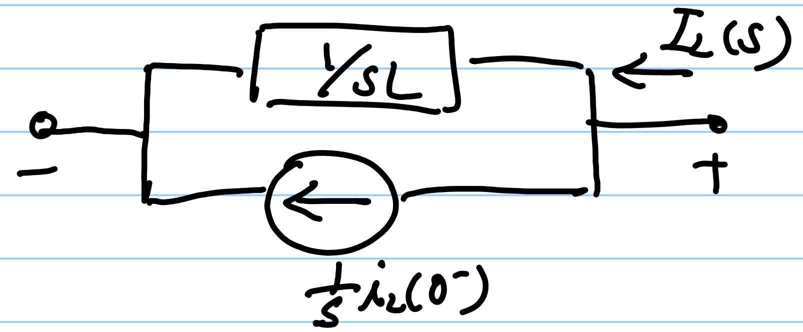
if $i_L(0) = 0 \Rightarrow V_L(s) = sL\,I_L(s)$ or $\dfrac{V_L(s)}{I_L(s)} = sL$

If not

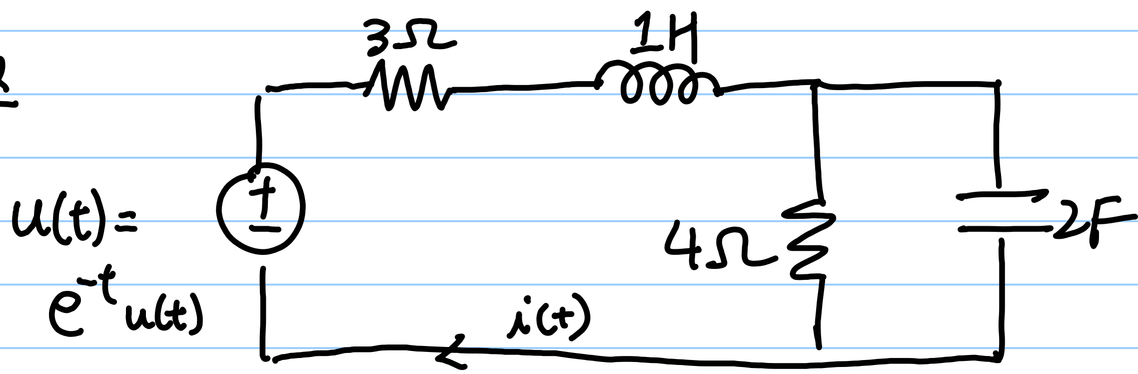
$$V_L(s) = sL\,I_L(s) - L\,i_L(0^-)$$



$$I_L(s) = \frac{1}{sL} V_L(s) + \frac{1}{s} i_L(0^-)$$

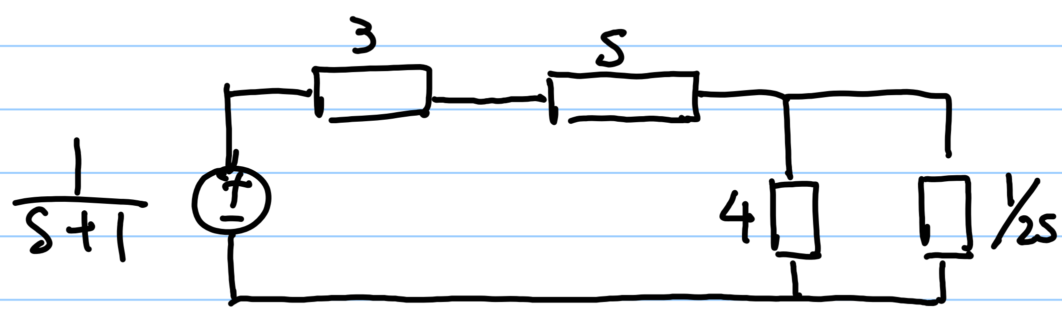


Example

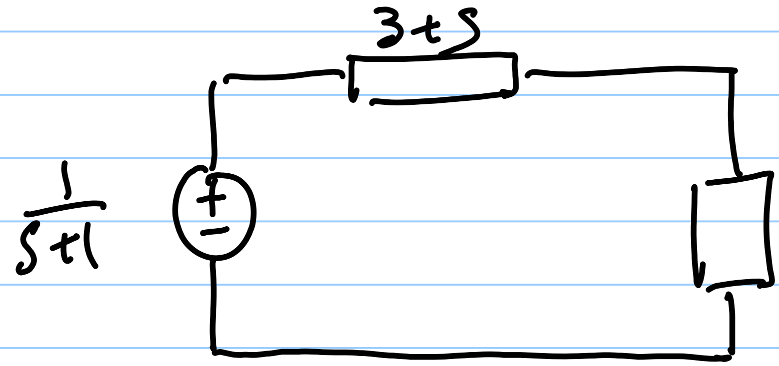


Find $i(t)$ assume Initial Conditions are zero.

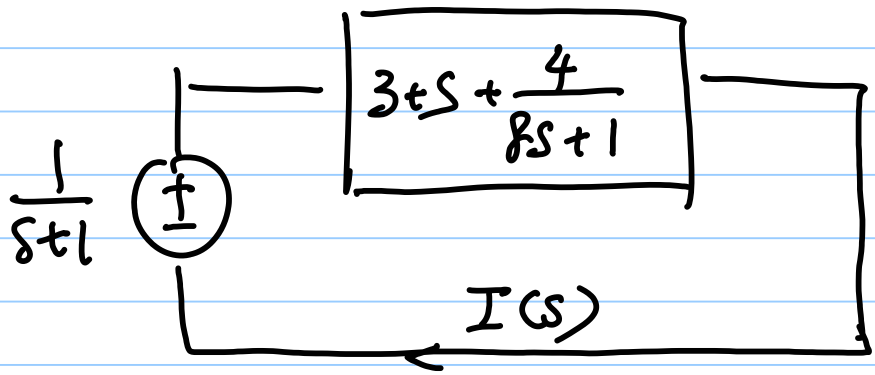
step 1. Convert the circuit to s-domain



step2 Treat as a resistive network



$$\frac{4 \cdot \frac{1}{2s}}{4 + \frac{1}{2s}} = \frac{2}{s\left[\frac{8s+1}{2s}\right]} = \frac{4}{8s+1}$$



$$I(s) = \frac{1/s+1}{\left(3+s+\frac{4}{8s+1}\right)}$$

Rational function

$\Rightarrow$ will talk about $\mathcal{L}^{-1}[\;]$ of that in a moment

$\Rightarrow \mathcal{L}^{-1}[I(s)] = i(t)$

Formula: $\cancel{x(t) = \frac{1}{2\pi j}\int_{c-2\pi j}^{c+2\pi j} X(s)e^{st}\,ds}$

Inverse LT

e.x. Find $\mathcal{L}^{-1}[X(s)]$ where

$$X(s) = \frac{s^5 + 5s^4 + 9s^3 + 9s^2 + 12s + 4}{s^4 + 4s^3 + 5s^2 + 4s + 4}$$

<u>Step 1</u> If $X(s)$ is not a Proper rational function, perform a long division.

Proper Rational function $\overset{\text{def}}{=}$

degree (Numeral polynomial) $<$ degree (Denominator)

$\frac{100}{23} = \frac{1\times10^2 + 0\times10^1 + 0\times10^0}{2\times10^1 + 3\times10^0}$ $\quad 5 \qquad\qquad 4$

$$\begin{array}{r|l}
 & s + 1 \\
\hline
s^4+4s^3+5s^2+4s+4 & s^5 + 5s^4 + 9s^3 + 9s^2 + 12s + 4 \\
 & -)\ s^5 + 4s^4 + 5s^3 + 4s^2 + 4s \\
\hline
 & s^4 + 4s^3 + 5s^2 + 8s + 4 \\
 & s^4 + 4s^3 + 5s^2 + 4s + 4 \\
\hline
 & 4s
\end{array}$$

$$\frac{100}{23} = 4\frac{8}{23}$$

$$X(s) = s+1 + \frac{4s}{s^4+4s^3+5s^2+4s+4}$$

proper!

$$\mathcal{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0^-)$$

$$\mathcal{L}^{-1}[1] = \delta(t) \qquad \mathcal{L}^{-1}[s] = \delta^{(1)}(t)$$

Rarely seen.

2°) Factorize the denominator. (MATLAB)

$$s^4+4s^3+5s^2+4s+4 = (s+2)^2(s^2+1) = (s+2)^2(s+j)(s-j)$$

In matlab >> roots([1 4 5 4 4])

deg = 1

3°) Partial Fraction Expansion

$$X(s) = \frac{4s}{(s+2)^2(s^2+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+j} + \frac{\overline{C}}{s-j}$$

unknown.

Four simple rules:

① $$\frac{\cdots\cdots}{\cdots(s+a)\cdots} \longrightarrow \cdots + \frac{A}{s+a} + \cdots$$

Real #

multiplicate root ② $$\frac{\cdots\cdots}{\cdots(s+a)^n\cdots} \longrightarrow \cdots + \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_n}{(s+a)^n} + \cdots$$

③ $$\frac{\cdots}{\cdots(s+\alpha)(s+\bar{\alpha})\cdots} \longrightarrow \cdots + \frac{\overset{\text{complex}}{A}}{s+\alpha} + \frac{\bar{A}}{s+\bar{\alpha}} + \cdots$$

conjugates

complex

conjugate ∵ real coefficients.

④ $$\frac{}{\cdots(s+\alpha)^n(s+\bar{\alpha})^n\cdots} \longrightarrow \cdots + \frac{A_1}{s+\alpha} + \frac{A_2}{(s+\alpha)^2} + \cdots + \frac{A_n}{(s+\alpha)^n} + \cdots$$
$$\cdots + \frac{\bar{A}_1}{s+\bar{\alpha}} + \frac{\bar{A}_2}{(s+\bar{\alpha})^2} + \cdots + \frac{\bar{A}_n}{(s+\bar{\alpha})^n} + \cdots$$

step 4) Solve for the unknowns in partial fraction

$$X(s) = \frac{4s}{(s+2)^2(s^2+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+j} + \frac{\bar{C}}{s-j}$$

$$\frac{\overbrace{(A+2Re(C))}^{=0}s^3 + \overbrace{(2A+B+8Re(C)+2Im(C))}^{=0}s^2 + \overbrace{(A+8Re(C)+8Im(C))}^{=4}s + \overbrace{(2A+B+8Im(C))}^{=0}}{(s+2)^2(s^2+1)}$$

Heaviside's Theorem:

$$X(s) = \frac{4s}{(s+2)^2(s^2+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+j} + \frac{\bar{C}}{s-j}$$

$$X(s)(s+2)^2 = A(s+2) + B + \frac{C(s+2)^2}{s+j} + \frac{\bar{C}(s+2)^2}{s-j}$$

$$\parallel$$

$$\frac{4s}{(s^2+1)}$$