EE422G Homework #10 Solution

1. Solution

(a)

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(nT - kT) z^{-n} = z^{-k}, \quad for\ all\ z$$

(b)

$$X(z) = \sum_{n=0}^{4} \left(\frac{1}{4}\right)^n z^{-n} = \sum_{n=0}^{4} \left(\frac{1}{4} z^{-1}\right)^n$$

which is

$$X(z) = \frac{1 - \left(\frac{1}{4}z^{-1}\right)^5}{1 - \frac{1}{4}z^{-1}} = \frac{z^5 - \frac{1}{1024}}{z^4\left(z - \frac{1}{4}\right)}$$

It looks like it has a pole at $z = \dfrac{1}{4}$. Note if we write $X(z)$ with form

$$X(z) = 1 + \frac{1}{4}\left(\frac{1}{z}\right) + \frac{1}{16}\left(\frac{1}{z}\right)^2 + \cdots + \frac{1}{256}\left(\frac{1}{z}\right)^4$$

We can see there is no pole at $z = \dfrac{1}{4}$ and there is a 4^{th} order pole at $z = 0$.

(c) Note the time convolution property of z-transform: $f_1[n] * f_2[n] \Rightarrow F_1(z) \cdot F_2(z)$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \cdot \frac{1}{1 - 2z^{-1}} z^{-n}, \quad |z| > 2$$

(d) Let

$$f(n) = \sin(\pi n/2) u(nT)$$

$$F(z) = \sum_{n=0}^{\infty} \sin(\pi n/2) z^{-n} = \frac{\sin(\pi/2) z^{-1}}{1 - 2\cos(\pi/2) z^{-1} + z^{-2}} = \frac{z^{-1}}{1 + z^{-2}}$$

From the multiplication by n property

$$n f[n] \Leftrightarrow -z \frac{d}{dz} F(z)$$

$$-z\frac{d}{dz}F(z) = -z\frac{d}{dz}\left(\frac{z^{-1}}{1 + z^{-2}}\right) = \frac{z^3 - z}{\left(1 + z^2\right)^2}$$

2. Solution

(a)

$$h[n] = \delta(n) + 2\delta(n-6) + 4\delta(n-8)$$

(b)

$$h[n] = \left(\frac{1}{2}\right)^{n-1} - \left(-\frac{1}{3}\right)^{n}$$

(c)

$$H(z) = \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$h[n] = \left(\frac{1}{2}\right)^{n-1} u[n]$$

(d) We know that

$$f[n] = a^n u[n] \Leftrightarrow F(z) = \frac{z}{z-a} \tag{1}$$

where $a = 3$ in this question. Taking the second derivative of the Eq. 1 with respect to z, we get

$$\frac{d^2}{dz^2}F(z) = \frac{d^2}{dz^2}\left(\frac{z}{z-a}\right) = \frac{d}{dz}\left[\frac{d}{dz}\left(\frac{z}{z-a}\right)\right] = \frac{d}{dz}\left[\frac{-a}{(z-a)^2}\right] = \frac{2a}{(z-a)^3} \tag{2}$$

also, from the multiplication by n^2 property

$$n^2 f[n] = n^2\left(a^n u[n]\right) \Leftrightarrow z\frac{d}{dz}F(z) + z^2\frac{d^2}{dz^2}F(z) \tag{3}$$

we know that

$$z\frac{d}{dz}F(z) = \frac{-az}{(z-a)^2} \tag{4}$$

and by substitution of (2) and (4) into (3) we get

$$n^2\left(a^n u[n]\right) \Leftrightarrow \frac{-az}{(z-a)^2} + \frac{2a}{(z-a)^3} = \frac{az(z+a)}{(z-a)^3}$$

We observe that for $a = 3$ the above reduces to

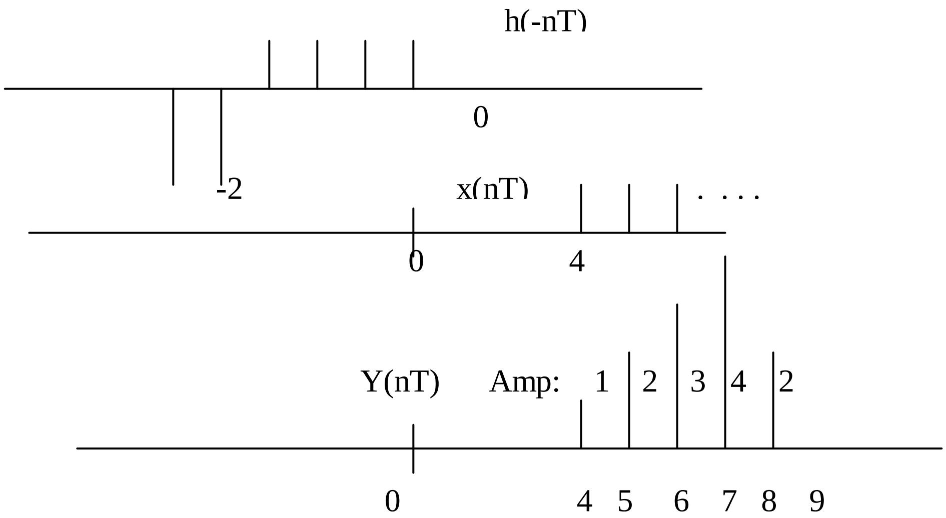
$$n^2 3^n u[n] \Leftrightarrow \frac{3z(z+3)}{(z-3)^3}$$

$$H(z) = Z(z)^2 = \frac{9z^2(z+3)^2}{(z-3)^6}$$

$$h[n] = -\frac{1}{30}n3^n + \frac{1}{30}n^5 3^n$$

There are other ways to solve this question.

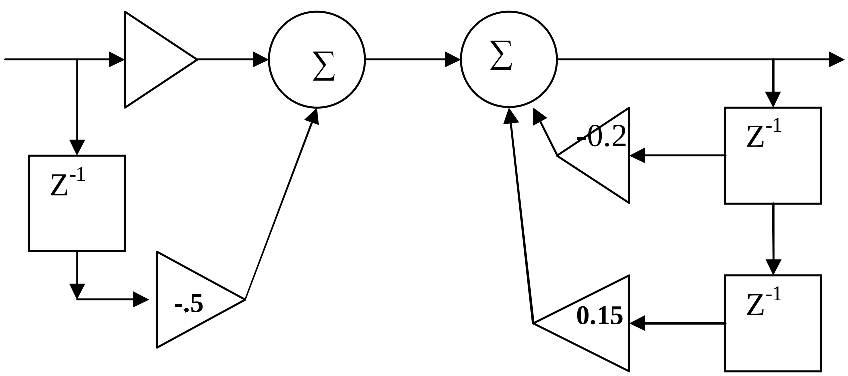
3. Solution



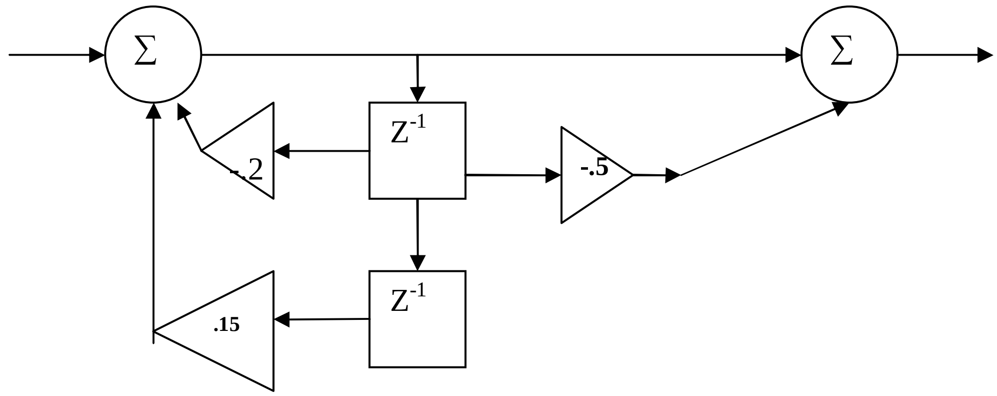
4. Solution

$$H(z)=\frac{1-0.5z^{-1}}{(1-0.3z^{-1})(1+0.5z^{-1})}$$

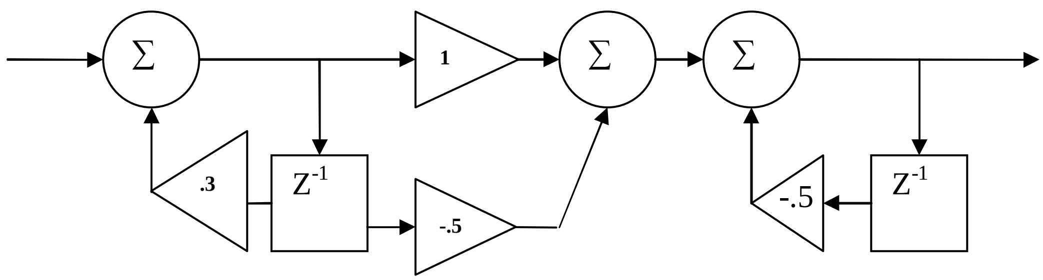
Direct Form I



Direct Form II

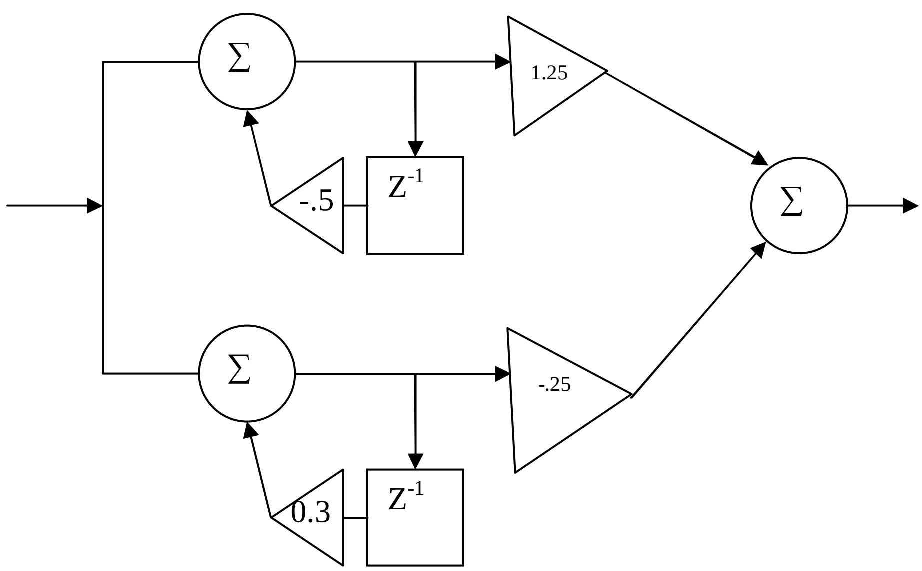


Cascade form:



Parallel form:

$H(Z) = 1.25/(1+0.5Z^{-1}) - 0.25/(1-0.3Z^{-1})$



5. Solution

(a)

$$h(nT) = (1/2)^n u(nT) \Leftrightarrow H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$y(nT) = 2\delta(nT-4T) \Leftrightarrow Y(z) = 2z^{-4}$$

$$X(z) = \frac{Y(z)}{H(z)} = \frac{2z^{-4}}{\frac{1}{1-\frac{1}{2}z^{-1}}} = 2z^{-4} - z^{-5}$$

$$x[n] = 2\delta[n-4] - \delta[n-5]$$

(b)

$$X(z) = 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}$$

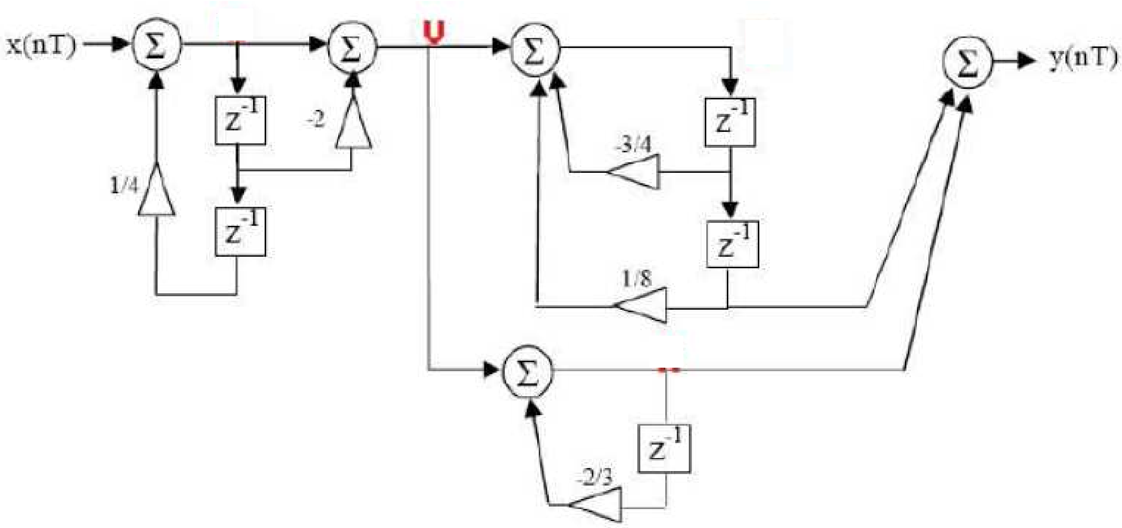
$$Y(z) = 1 - \frac{3}{4}z^{-1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{3}{4}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

The transfer function is

$$h[n] = -\frac{2}{3}\left(\frac{1}{4}\right)^n + \frac{3}{5}\left(-\frac{1}{2}\right)^n$$

6. Solution



The component between the intermediate output v(nT) and the input x(nT) is a direct form II implementation of the following transfer function

$$H_1(z) = \frac{-2z^{-1}+1}{1-\frac{1}{4}z^{-2}}$$

Between v(nT) and the output y(nT), it is a parallel form with two branches:

$$H_2(z) = \frac{z^{-2}}{1 + \frac{3}{4} z^{-1} - \frac{1}{8} z^{-2}} + \frac{1}{1 + \frac{2}{3} z^{-1}}$$

The overall system is simply $H_1(z)H_2(z)$