1. (2 points) A discrete-time filter has the unit pulse response 
\[ h(nT) = 4[u(n) - u(n - 12)] \]. Place the frequency response in the form 
\[ H(e^{j\omega T}) = K \frac{A(\omega)}{\sin B(\omega)} e^{jC(\omega)} \]
by determining \( A(\omega), B(\omega), C(\omega) \), and \( K \). Assuming a sampling frequency of 1000 Hz, determine the amplitude and phase responses of the filter at \( f = 0, 25, \) and 50 Hz.

By taking the Z-transform of the impulse response, we can obtain the transfer function as
\[ H(z) = 4 \left( \frac{1}{1 - z^{-12}} - \frac{z^{-12}}{1 - z^{-1}} \right) = 4(1 - z^{-12}) \]

To evaluate the DTFT, we evaluate the above Z-transform at \( z = e^{j\omega T} \) to obtain:
\[ H(e^{j\omega T}) = \frac{4(1 - e^{-j12\omega T})}{1 - e^{-j\omega T}} \]
\[ = \frac{4e^{-j6\omega T}(e^{j6\omega T} - e^{-j6\omega T})}{e^{-j\omega T/2}(e^{j\omega T/2} - e^{-j\omega T/2})} \]
\[ = \frac{4e^{-j6\omega T} \sin(6\omega T)}{\sin(\omega T)} = 4 \frac{\sin(6\omega T)}{\sin(\omega T)} e^{-j12\omega T} \]

Clearly, the second step above is the key to arrive at the two sine functions. Using the above expression and substituting \( T = 1/1000 \) and \( \omega = 2\pi(25), \) and \( 2\pi(50) \), we obtain

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>Magnitude Response</th>
<th>Phase Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>50\pi</td>
<td>41.2453</td>
<td>-1.0210</td>
</tr>
<tr>
<td>100\pi</td>
<td>24.3183</td>
<td>-2.0420</td>
</tr>
</tbody>
</table>

Direct evaluation of \( \omega = 0 \) will result in 0/0. The transfer function appears to have a pole at 0. However, it is actually not true because the pole is actually canceled out by a zero:

\[
> \text{syms } z \\
> \text{simplify(4*(1-z^(-12))/(1-z^-1))}
\]

\[ \text{ans} = 4/z^{11}*(z^{11}+z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1) \]

i.e. \( H(z) = 4(1+z^{-1}+z^{-2}+z^{-3}+z^{-4}+z^{-5}+z^{-6}+z^{-7}+z^{-8}+z^{-9}+z^{-10}+z^{-11}) \)

This is in fact a FIR! And \( H(exp(0)) = H(1) = 48. \)

2. (2 points) Compute the 4-point DFT of
\[ x(n) = \exp\left(\frac{jn\pi}{4}\right), \quad n = 0,1,2,3 \]

\[ X(0) = x(0) + x(1) + x(2) + x(3) = 1 + j2.4142 \]

\[ X(1) = x(0) + x(1)\exp\left(\frac{j2\pi}{4}\right) + x(2)\exp\left(\frac{j4\pi}{4}\right) + x(3)\exp\left(\frac{j6\pi}{4}\right) = 1 - j2.4142 \]

\[ X(2) = x(0) + x(1)\exp\left(\frac{j4\pi}{4}\right) + x(2)\exp\left(\frac{j8\pi}{4}\right) + x(3)\exp\left(\frac{j12\pi}{4}\right) = 1 - j0.4142 \]

\[ X(3) = x(0) + x(1)\exp\left(\frac{j6\pi}{4}\right) + x(2)\exp\left(\frac{j12\pi}{4}\right) + x(3)\exp\left(\frac{j18\pi}{4}\right) = 1 + j0.4142 \]

3. (2 points) Perform the 6-point circular convolution of the following signal with \( u(nT-4T) \). Compare your answer with that from Problem 3 of HW 10.

4. (4 points) Signal Space
Many of you are familiar with the concept of an inner-product space – a vector space that comes with an inner (dot) product operation. Given two vectors \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) and \( \mathbf{w} = (w_1, w_2, \ldots, w_n) \), their inner product is given by
\[
\langle \mathbf{v}, \mathbf{w} \rangle := \sum_{i=1}^{n} v_i w_i
\] (1)
Note the conjugate sign on the components \( w_i \) – this is needed if one allows vectors with complex coefficients. The most useful feature of an inner-product space is the existence of an “orthonormal” basis – a finite set of unit-length vectors that are orthogonal with each other and span the entire space. In another words, if \( \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \} \) is an orthonormal basis, we have
\[
\begin{align*}
\langle \mathbf{e}_i, \mathbf{e}_i \rangle &= 1 \text{ for all } i \\
\langle \mathbf{e}_i, \mathbf{e}_j \rangle &= 0 \text{ for all } i,j \text{ with } i \neq j
\end{align*}
\]
Every vector \( \mathbf{x} \) in the vector can then be written as a linear combination of these vectors with the coefficients being the inner-product between \( \mathbf{x} \) and \( \mathbf{e}_i \):
\[
\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{e}_i \rangle \mathbf{e}_i
\] (2)
The simplest example will be the three unit vectors along x-, y- and z-axes in the three-dimensional Euclidean space. It turns out that we can also interpret the N-point DFT as a representation along orthonormal basis. The vector space, or signal space, is all the signals with N samples: \( \mathbf{x} = (x(0), \ldots, x(N-1)) \). The orthonormal basis are \( \{ \mathbf{e}_i, \mathbf{e}_i, \ldots, \mathbf{e}_i \} \) where
\[
\mathbf{e}_i = \left( \frac{1}{\sqrt{N}} \exp \left( \frac{j2\pi (i-1)n}{N} \right) : n = 0, \ldots, N-1 \right)
\]
(a) Show that \( \langle \mathbf{x}, \mathbf{e}_i \rangle \) corresponds to the analysis formula of the \( (i-1) \)th DFT coefficient. Just the definition (except for the difference of a scalar factor of one over root N):
\[
\langle \mathbf{x}, \mathbf{e}_i \rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \exp \left( -\frac{j2\pi (i-1)n}{N} \right) = \frac{1}{\sqrt{N}} \mathcal{X}(i-1)
\]
b) Show that equation (2) in the signal space corresponds to the synthesis formula of DFT. Again, just the definition:
\[
\sum_{n=1}^{N} \langle \mathbf{x}, \mathbf{e}_i \rangle \mathbf{e}_j = \sum_{n=1}^{N} \frac{1}{\sqrt{N}} \mathcal{X}(i-1) \frac{1}{\sqrt{N}} \exp \left( -\frac{j2\pi (i-1)n}{N} \right) = \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{X}(i) \exp \left( -\frac{j2\pi in}{N} \right) = x(n)
\]
c) Prove that the set \( \{ \mathbf{e}_i, i=1, \ldots, N \} \) as defined above is orthonormal. We need to show that they satisfy two properties. First,
\[
\langle \mathbf{e}_i, \mathbf{e}_i \rangle = \frac{1}{N} \sum_{n=1}^{N} \exp \left( \frac{j2\pi (i-1)n}{N} \right) \exp \left( -\frac{j2\pi (i-1)n}{N} \right) = \frac{1}{N} \sum_{n=1}^{N} 1 = \frac{1}{N} N = 1
\]
Second, we have \( \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \frac{1}{N} \sum_{n=1}^{N} \exp \left( -\frac{j2\pi n}{N} (j-i) \right) \) which is the \( (j-i) \)th DFT coefficient of the unit step sequence \( u(n) \). Recall that the DFT of the unit step sequence is the delta function \( \delta(k) \). Thus \( \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta(i-j) \).