EE422G Homework #14(12 points)

1. Linear Algebra:

a. Find the determinant of $A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & -4 \end{pmatrix}$

$$\det(A) = -1 \cdot \det\begin{pmatrix} 3 & 0 \\ 1 & -4 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = 12$$

b. Find the inverse of A.

$$A^{-1} = \frac{1}{12}\begin{pmatrix} -12 & 0 & 0 \\ 4 & 4 & 0 \\ 1 & 1 & -3 \end{pmatrix}$$

2. (2 points) Identify the state matrices A, B, C, D AND the transfer function H(s) for the following set of equations

$$\begin{aligned} \dot{x}_1 &= -4x_1 + 3x_2 + 6u \\ \dot{x}_2 &= -x_1 - 7x_2 - 4u \\ y &= 5x_1 - 3x_2 + 2u \end{aligned}$$

$$A = \begin{pmatrix} -4 & 3 \\ -1 & -7 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ -4 \end{pmatrix} \quad C = \begin{pmatrix} 5 & -3 \end{pmatrix} \quad D = 2$$

$$H(s) = C(sI - A)^{-1}B + D$$

$$= \begin{pmatrix} 5 & -3 \end{pmatrix}\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -4 & 3 \\ -1 & -7 \end{pmatrix}\right]^{-1}\begin{pmatrix} 6 \\ -4 \end{pmatrix} + 2$$

$$= \begin{pmatrix} 5 & -3 \end{pmatrix}\begin{pmatrix} s+4 & -3 \\ 1 & s+7 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ -4 \end{pmatrix} + 2$$

$$= \frac{1}{(s+4)(s+7)+3}\begin{pmatrix} 5 & -3 \end{pmatrix}\begin{pmatrix} s+7 & 3 \\ -1 & s+4 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \end{pmatrix} + 2$$

$$= \frac{2s^2 + 64s + 278}{s^2 + 11s + 31}$$

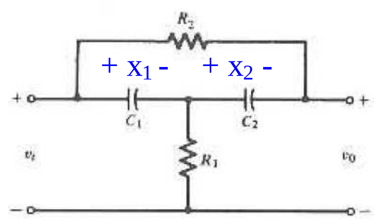
3. Obtain a state model for the followings:

a. $H(s) = \dfrac{s+1}{s^2+s+3}$

We will use the approach introduced during the lecture. The dimension of the state vector is 2 as the degree of the denominator polynomial is 2. Since the leading coefficient of the denominator is already 1, no need to normalize. We can proceed to the state equations directly:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -3 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u$$

$$y = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$



b.

Define the state x_1=voltage across the capacitor C_1

x_2=voltage across the capacitor C_2

Then, the current across the capacitor C_1 = $i_1 = C_1\dot{x}_1$

The current across the capacitor C_2 = $i_2 = C_2\dot{x}_2$

Let's write the state equation that relates the output voltage with the input voltage and the state. Observe that the voltage across the resistor R_2 is just the sum of the two capacitor voltages, i.e. x_1+x_2. Thus, apply KVL around the outmost loop does the job:

$$v_0 = -x_1 - x_2 + v_i$$

Next, we need to derive the dynamics equation which relates $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}$ with the state vector $\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ and the input. There are two loops that do not involve the output voltage – the one containing R_2, C_1 and C_2 and the one containing v_i, C_1 and R_1. KVL around the first loop gives

$$-i_2R_2 = x_1 + x_2$$

$$\Rightarrow -\dot{x}_2C_2R_2 = x_1 + x_2$$

$$\Rightarrow \dot{x}_2 = {}^{-1}\!/_{C_2R_2}\, x_1 - {}^{1}\!/_{C_2R_2}\, x_2$$

KVL around the second loop gives

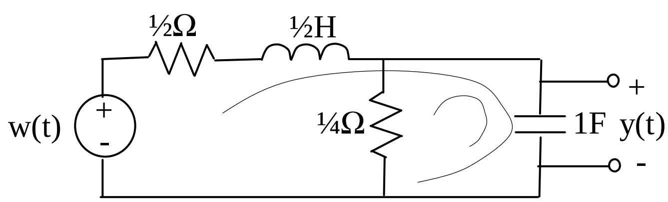
$$x_1 + (i_1 - i_2)R_1 = v_i$$
$$\Rightarrow x_1 + (C_1\dot{x}_1 - C_2\dot{x}_2)R_1 = v_i$$
$$\Rightarrow x_1 + [C_1\dot{x}_1 - C_2(\tfrac{-1}{C_2R_2}x_1 - \tfrac{1}{C_2R_2}x_2)]R_1 = v_i$$
$$\Rightarrow \dot{x}_1 = (\tfrac{-1}{C_1R_2} - \tfrac{1}{C_1R_1})x_1 - \tfrac{1}{C_1R_2}x_2 + \tfrac{1}{C_1R_1}v_i$$

Putting them altogether in the standard form:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}\tfrac{-1}{C_1R_2} - \tfrac{1}{C_1R_1} & \tfrac{-1}{C_1R_1}\\ \tfrac{-1}{C_2R_2} & \tfrac{-1}{C_2R_2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}\tfrac{1}{C_1R_1}\\ 0\end{pmatrix}v_i$$

$$v_0 = \begin{pmatrix}-1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + v_i$$

4. (4 points) Given the following circuit



a. Define an appropriate state vector for the above circuit.

Since there are two memory-storage elements, the state vector is two-dimensional:

x_1 = voltage the capacitor

x_2 = current through the inductor

b. Write the output equation in the form of $y = Cx + Dw$.

The output is just the voltage across the capacitor, which is one of our states:

$$y = x_1$$
$$y = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

c. Write the dynamics equation in the form of $\dot{x} = Ax + Bw$. Hint: Use KVL on the two loops indicated in the drawing.

Applying KVL around the bigger loop: $\dot{x}_2 = -2x_1 - x_2 + 2w$

Applying KVL around the smaller loop: $\dot{x}_1 = -4x_1 + x_2$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-4 & 1\\ -2 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 2\end{pmatrix}w$$