EE422G Homework #3 Solution

1. (2 points)

   (a) By definition, we have

$$
\begin{aligned}
L\left[\int_{-\infty}^{t} x(\tau)d\tau\right] &= \int_{0}^{\infty}\int_{-\infty}^{t} x(\tau)d\tau e^{-st}dt \\
&= \int_{0}^{\infty}\int_{-\infty}^{t} x(\tau)d\tau d\left(\frac{-e^{-st}}{s}\right)
\end{aligned}
$$

   Again, we will use integration by parts:

$$
\int_{b}^{a} u(t)dv(t) = u(t)v(t)|_{t=a}^{t=b} - \int_{a}^{b} v(t)du(t)
$$

   Make the following substitutions:

$$
\begin{aligned}
\frac{-e^{-st}}{s} &\Rightarrow v(t) \\
\int_{-\infty}^{t} x(\tau)d\tau &\Rightarrow u(t)
\end{aligned}
$$

   Then,

$$
\begin{aligned}
L\left[\int_{-\infty}^{t} x(\tau)d\tau\right] &= \int_{0}^{\infty}\int_{-\infty}^{t} x(\tau)d\tau d\left(\frac{-e^{-st}}{s}\right) \\
&= \left(\int_{-\infty}^{t} x(\tau)d\tau\right)\frac{-e^{-st}}{s}\Bigg|_{t=0}^{t=\infty} - \int_{0}^{\infty}\frac{-e^{-st}}{s}d\left(\int_{-\infty}^{t} x(\tau)d\tau\right) \\
&= \lim_{t\to\infty}\frac{-e^{-st}}{s}\int_{\infty}^{t} x(\tau)d\tau + \frac{e^{-s0}}{s}\int_{-\infty}^{0} x(\tau)d\tau - \int_{0}^{\infty}\frac{e^{-st}}{s}x(t)dt \\
&= 0 + \frac{y(0)}{s} - \frac{X(s)}{s}
\end{aligned}
$$

   (b) For simplicity, we assume that all the initial conditions are zero. If you assume otherwise, your answer may be different.

$$
\begin{aligned}
L^{-1}\left[s^2Y(s) + 2sY(s)\right] &= L^{-1}\left[s^3X(s) + \frac{X(s)}{s} - \frac{e^{-s}}{s}\right] \\
\frac{d^2y}{dt^2} + 2\frac{dy}{dt} &= \frac{d^3x}{dt^3} + \int_{-\infty}^{t} x(\tau)d\tau - u(t-1)
\end{aligned}
$$

2. (2 points) The strategy is to first apply Laplace Transform to both differential equations turning them into algebraic equations involving $X(s)$ and $Y(s)$, then solve for $X(s)$ and $Y(s)$ and finally convert them back to time domain by applying inverse Laplace Transform.

$sX(x) - x(0^-) + 3X(s) + 2Y(s) = \frac{1}{s}$ with $x(0) = 0$, we get

$$(s+3)X(s) + 2Y(s) = \frac{1}{s}$$

$sY(s) - y(0^-) - X(s) = 0$ with $y(0) = 0$, we get

$$sY(s) - X(s) = 0$$

Solving for $X(s)$ and $Y(s)$, we get

$$X(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

$$Y(s) = \frac{1}{s^3 + 3s^2 + 2s} = \frac{1}{s(s+1)(s+2)}$$

To apply the inverse Laplace Transform, we expand $X(s)$ and $Y(s)$ into partial fractions:

$$X(s) = \frac{a}{s+1} + \frac{b}{s+2}$$

$$Y(s) = \frac{c}{s} + \frac{d}{s+1} + \frac{e}{s+2}$$

Heaviside's theorem allows us to compute a, b, c, d, and e:

$$a = (X(s)(s+1))\,|_{s=-1} = 1$$

$$b = (X(s)(s+2))\,|_{s=1} = -1$$

$$c = (Y(s)s)\,|_{s=0} = 1/2$$

$$d = (Y(s)(s+1))\,|_{s=-1} = -1$$

$$e = (Y(s)(s+2))\,|_{s=-2} = 1/2$$

Thus, we have

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$Y(s) = \frac{1}{2s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

Taking inverse Laplace Transform will result in

$$x(t) = e^{-t} - e^{-2t}$$

$$y(t) = \frac{1}{2}u(t) - e^{-t} + \frac{1}{2}e^{-2t}$$

I will not take off any points if you use MATLAB to perform the inverse Laplace transform.

3. (2 points) The important observation is that the derivative of the triangular function can be easily expressed in terms of unit step function:

$$\frac{d}{dt}x(t) = u(t) - 2u(t-1) + u(t-2)$$

Taking the Laplace transform on both sides, we obtain the following based on the differentiation theorem and the time-delay property:

$$sX(s) - x(0^-) = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$

Since $x(0^-) = 0$, we have

$$X(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$

4. (4 points)

(a)

$$X(s) = \frac{5}{s^3 + s^2 + 9s + 9}$$

Initial value:

$$x(0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{5s}{s^3 + s^2 + 9s + 9} = 0$$

Final value: Since the poles of $sX(s)$ are 3i, -3i and -1, its ROC DOES NOT CONTAIN THE IMAGINARY AXIS, we cannot use the final value theorem. Using matlab to carry out the inverse Laplace transform, we find out that

$$x(t) = -\frac{1}{2}\cos(3t) + \frac{1}{6}\sin(3t) + \frac{1}{2}e^{-t}$$

whose final value clearly does not exist due to the oscillating sinusoids.

(b)

$$X(s) = \frac{s^2 + 5s + 7}{s^2 + 3s + 2}$$

Initial value:

$$x(0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{s^3 + 5s^2 + 7s}{s^2 + 3s + 2} \quad \textit{does not exist}$$

Final value: Since the poles of $sX(s)$ are -1 and -2, the ROC contains the imaginary axis and we can apply the final value theorem:

$$x(\infty) = \lim_{s \to 0} sX(s) = 0$$

5. (2 points)

(a)

```
>> y1 = ilaplace( sym('(s^3+6*s^2+11*s+6)/(s^2+4)^2') )

y1 =

(9/4*t+1)*cos(2*t)+(15/8+7/4*t)*sin(2*t)


>> y2 = ilaplace( sym('(s^3+6*s^2+11*s+6)/(s^2+4*s+4)') )

y2 =

dirac(1,t)+2*dirac(t)-exp(-2*t)


>> x = y1*y2

x =

((9/4*t+1)*cos(2*t)+(15/8+7/4*t)*sin(2*t))*(dirac(1,t)+2*dirac(t)-exp(-2*t))
```

(b)

```
>> den = [1 0 8 0 16];
>> num = [1 6 11 6];
>> [r,p,k] = residue(num, den)

r =

   0.5000 - 0.9375i
   1.1250 - 0.8750i
   0.5000 + 0.9375i
   1.1250 + 0.8750i


p =

  -0.0000 + 2.0000i
  -0.0000 + 2.0000i
  -0.0000 - 2.0000i
  -0.0000 - 2.0000i


k =

     []
===============================
```

```
>> den = [1 4 4];
>> num = [1 6 11 6];
>> [r,p,k] = residue(num, den)

r =

    -1
     0


p =

    -2
    -2


k =

     1     2
```