

## EE422G Homework #4 Solution (14 Points)

1. (3 points)

(a) (1 point) Proof of the FVT: Starting from the right hand side, we have

$$\begin{aligned}
 \lim_{s \rightarrow 0} sX(s) &= \lim_{s \rightarrow 0} L \left[ \frac{dx}{dt} \right] + x(0^-) \\
 &= \lim_{s \rightarrow 0} \int_0^\infty \frac{dx}{dt} e^{-st} dt + x(0^-) \\
 &= \int_0^\infty \frac{dx}{dt} dt + x(0^-) \\
 &= [x(t)]_{0^-}^\infty + x(0^-) \\
 &= \lim_{t \rightarrow \infty} x(t) - x(0^-) + x(0^-) \\
 &= \lim_{t \rightarrow \infty} x(t)
 \end{aligned}$$

(b) (2 points) To use FVT, we need to check if the ROC of  $sH(s)$  contains the imaginary axis.

i. The poles of  $sH(s)$  are  $-4$  and  $-2$ , which implies that the ROC is  $\{z : \operatorname{Re}(z) > -2\}$ . The ROC contains the imaginary axis and FVT can be used:

$$h(\infty) = \lim_{s \rightarrow 0} sH(s) = 0$$

ii. The poles of  $sH(s)$  are  $-2j, 2j$  and  $-3$ , which implies that the ROC is  $\{z : \operatorname{Re}(z) > 0\}$ . The ROC does not contain the imaginary axis and FVT cannot be used. In fact, you can do the inverse Laplace transform and get

$$h(t) = -\frac{2}{13} \cos(t)^2 + \frac{16}{13} \sin(t) \cos(t) + \frac{2}{13} e^{-3t/2} \cosh\left(\frac{3t}{2}\right)$$

The first two terms oscillate and do not converge to anything.

2. (2 points)

(a) Write  $X(s) = \frac{7s^2+15s+10}{(s+1)^2(s+3)}$  into partial fraction expansion:

$$X(s) = \frac{a}{(s+1)} + \frac{b}{(s+1)^2} + \frac{c}{(s+3)}$$

Based on Heaviside's theorem:

$$\begin{aligned}
 a &= \frac{d}{dt} ((s+1)^2 X(s)) \Big|_{s=-1} \\
 &= \frac{d}{dt} \left( \frac{7s^2 + 15s + 10}{s+3} \right) \Big|_{s=-1} \\
 &= 0 \\
 b &= ((s+1)^2 X(s)) \Big|_{s=-1} \\
 &= 1 \\
 c &= ((s+3)X(s)) \Big|_{s=-3} \\
 &= 7
 \end{aligned}$$

Thus, we have

$$X(s) = \frac{1}{(s+1)^2} + \frac{7}{(s+3)}$$

and the time-domain signal is

$$x(t) = te^{-t}u(t) + 7e^{-3t}u(t)$$

(b) Write  $X(s) = \frac{s^4+8s^2+s+17}{(s^2+4)^2(s+1)}$  into partial fraction expansion:

$$X(s) = \frac{a}{s+2i} + \frac{\bar{a}}{s-2i} + \frac{b}{(s+2i)^2} + \frac{\bar{b}}{(s-2i)^2} + \frac{c}{s+1}$$

Applying Heaviside's theorem, we get

$$\begin{aligned} a &= \left. \frac{d}{dt} (X(s)(s+2i)^2) \right|_{s=-2i} \\ &= 0.0313i \\ b &= \left. (X(s)(s-2i)^2) \right|_{s=2i} \\ &= -0.0625 \\ c &= \left. (X(s)(s+1)) \right|_{s=-1} \\ &= 1 \end{aligned}$$

Applying the inverse Laplace transform, we got

$$x(t) = 0.0313ie^{-2it} - 0.0313ie^{2it} - 0.0625te^{-2it} + 0.0625te^{2it} + e^{-t}$$

You can further simplify the expression to obtain:

$$x(t) = 0.0625 \sin(2t) - 0.0125t \cos(2t) + e^{-t}$$

3. (2 points)

(a) Writing down the partial fraction expansion, we have

$$\begin{aligned} X(s) &= \frac{2s^4 - 3s^3 + 36s^2 - 26s + 162}{s(s^2+9)^2} \\ &= \frac{a}{s} + \frac{b}{(s+3j)^2} + \frac{\bar{b}}{(s-3j)^2} + \frac{c}{s+3j} + \frac{\bar{c}}{s-3j} \end{aligned}$$

Apply Heaviside's theorem, we have:

$$\begin{aligned} a &= X(s)s|_{s=0} = 2 \\ b &= \left. X(s)(s+3j)^2 \right|_{s=-3j} = -0.0278 \\ c &= \left. \frac{d}{ds} X(s)(s+3j)^2 \right|_{s=-3j} = 0.4907j \end{aligned}$$

Applying termwise Laplace transform, we get

$$\begin{aligned} x(t) &= 2u(t) - 0.0278te^{-3jt} - 0.0278te^{3jt} + 0.4907je^{-3jt} - 0.4907je^{3jt} \\ &= 2u(t) - 0.0556t \cos(3t) + 0.9815 \sin(3t) \end{aligned}$$

(b) Using long division, expand the factor involving the exponential as

$$\frac{1}{1 + e^{-2s}} = 1 - e^{-2s} + e^{-4s} - e^{-6s} + \dots$$

Thus

$$X(s) = \frac{1}{s + 3} [1 - e^{-2s} + e^{-4s} - e^{-6s} + \dots]$$

Now use the transform pair for an exponential, superposition, and the delay theorem to get

$$x(t) = e^{-3t}u(t) - e^{-3(t-2)}u(t-2) + e^{-3(t-4)}u(t-4) - e^{-3(t-6)}u(t-6) + \dots$$

4. (2 points)

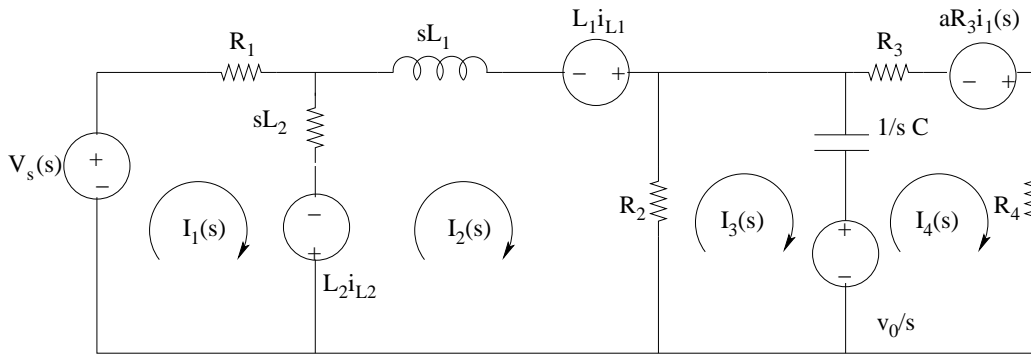


Figure 1: Problem 4.

5. (2 points) Replace the coupled coil with a T-equivalent circuit as shown in Fig. 2:

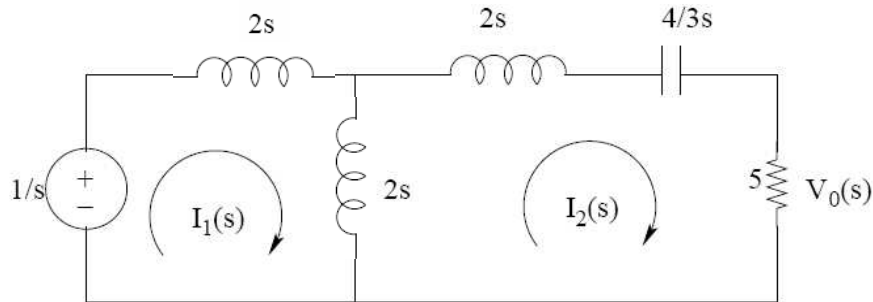


Figure 2: Problem 5.

Writing KVL equations around each loop, we obtain

$$\begin{bmatrix} 4s & -2s \\ -2s & 4s + \frac{4}{3s} + 5 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 1/s \\ 0 \end{bmatrix}$$

Solve for  $I_2(s)$  and then find the output voltage:

$$V_0(s) = 5I_2(s) = \frac{5}{6(s^2 + \frac{5}{3}s + \frac{4}{9})} = \frac{5}{6(s + 1/3)(s + 4/3)} = \frac{A}{s + 1/3} + \frac{B}{s + 4/3}$$

The expansion coefficient are

$$A = (s + 1/3)V_0(s) \Big|_{s=-1/3} = \frac{5}{6}$$

and

$$B = (s + 4/3)V_0(s) \Big|_{s=-4/3} = -\frac{5}{6}$$

Thus the output current is

$$v_0(t) = \frac{5}{6} \left[ e^{-t/3} - e^{-4t/3} \right] u(t)$$