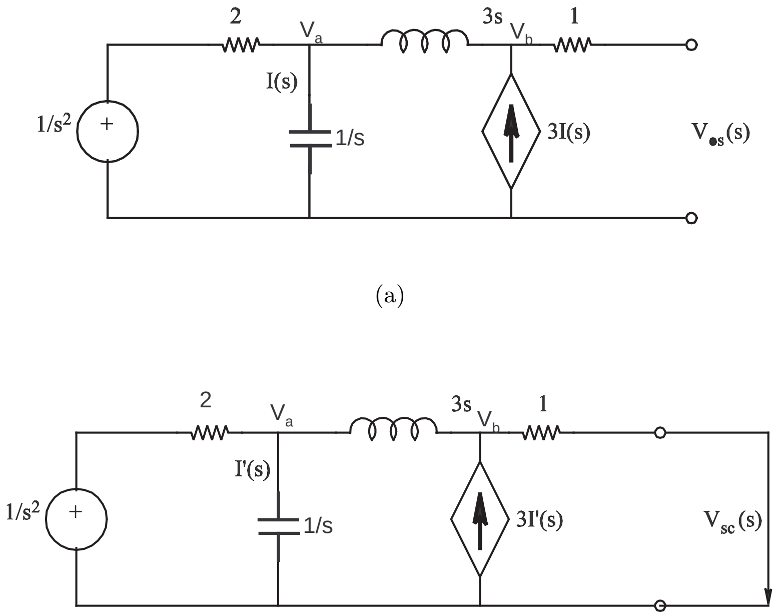
EE422G Homework #5 Solution

1. (2 points) Use the fact that

$$Z_{eq}(s) = V_{oc}(s)/I_{sc}(s)$$

The Fig. 1(a) may be used to find $V_{os}(s)$ and the Fig. 1(b) to find $I_{sc}(s)$.



(a)

(b)

Figure 1: Qutestion 1

The KCL equations for the first circuit are

$$\frac{1}{2}\left(V_a - \frac{1}{s^2}\right) + I + \frac{V_a - V_b}{3s} = 0$$

$$sV_a = I$$

$$\frac{V_b - V_a}{3s} = 3I = 3sV_a$$

Solve the third equation for V_b in terms of V_a

$$V_b = (9s^2 + 1)V_a$$

Substitute into the first equation to get

$$V_b = V_{oc} = \frac{(9s^2 + 1)}{s^2(1 - 4s)}$$

Now work with the second circuit to get the short circuit current. Note the all voltages and currents are different than for the first circuit. The node voltage equations are

$$\frac{1}{2}\left(V_a - \frac{1}{s^2}\right) + I' + \frac{V_a - V_b}{3s} = 0$$

$$I' = sV_a$$

$$\frac{V_b - V_a}{3s} - 3I' + V_b = 0$$

Solve the third for V_a in terms of V_b

$$V_a = \frac{1+3s}{1+9s^2}V_b$$

Substitute into the first equation and solve for V_b. Note that $I_{sc} = V_b/1$

$$I_{sc} = \frac{V_b}{1} = \frac{9s^2+1}{s^2(6s^2-s+3)}$$

Divide V_{oc} by I_{sc} to get Z_T

$$Z_T = \frac{6s^2-s+3}{1-4s}$$

2. (2 points)

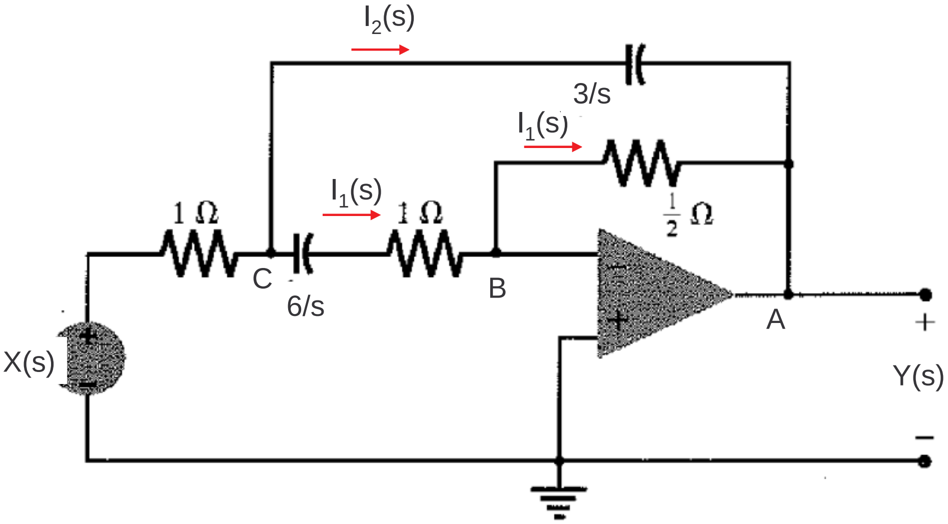


Figure 2: Qutestion 2

Figure 2 shows the circuit in Laplace domain. Since the non-inverting input of the op-amp is ground, the op-amp gain will be negative. To find the transfer function, we need to relate the output with the input $X(s)$. It makes sense then to apply KCL at node C as the left branch involves the input and the top branch involves the output. To find the current, we need to find the voltage V_C at node C. The ideal model of op-amp states that the voltage difference between the two input ports is zero. Thus, we have $V_B = 0$. The ideal model also states that the input current is zero. As a result, the current between nodes C and B must be the same as that between B and A. We call this current I_1, which can be easily expressed in terms of the output voltage $Y(s)$:

$$I_1(s) = \frac{-Y(s)}{1/2} = -2Y(s)$$

Subsequently, V_C can be computed as follows:

$$V_C(s) = I_1(s)(\frac{6}{s} + 1) = Y(s)\frac{-12 - 2s}{s}$$

Now, we are ready to apply KCL at node C:

$$\begin{aligned}
\frac{X(s) - V_C(s)}{1} &= I_1(s) + I_2(s) \\
X(s) - V_C(s) &= -2Y(s) + \frac{V_C(s) - Y(s)}{3/s} \\
3X(s) &= (-s-6)Y(s) + (s+3)V_C(s) \\
3X(s) &= (-s-6)Y(s) + \frac{(-12-2s)(s+3)}{s}Y(s) \\
\frac{Y(s)}{X(s)} &= \frac{s}{-s^2 - 8s - 12}
\end{aligned}$$

which is the transfer function of the circuit.

3. (4 pts)

   (a) Find the transfer function $H(s) = \frac{Y(s)}{W(s)}$ for the following circuit:

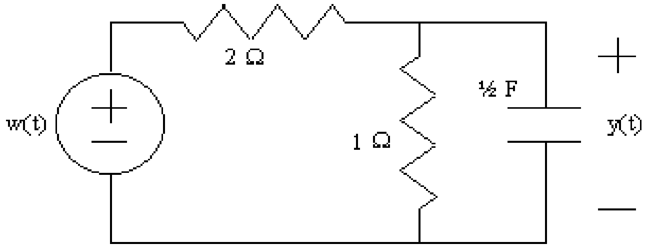


Figure 3: Problem 1a

This is just a simple voltage divider:

$$\begin{aligned}
H(s) &= \frac{Y(s)}{W(s)} \\
&= \frac{2/(s+2)}{2 + 2/(s+2)} \\
&= \frac{1}{s+3}
\end{aligned}$$

(b) A dynamic System is governed by the following differential equation with initial conditions $\left.\frac{dy}{dt}\right|_{t=0} = 1$ and $y(0) = 0$,

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = x(t) \tag{1}$$

Given the input $X(s) = 1$, write down the laplace transform of the zero-state response and the zero-input response.

Applying Laplace transform onto the differential equation, we have

$$s^2Y(s) - sy(0) - y^{(1)}(0) + 2sY(s) - 2y(0) + Y(s) = X(s)$$

or

$$Y(s) = \frac{X(s)}{s^2 + 2s + 1} + \frac{sy(0) + 2y(0) + y^{(1)}(0)}{s^2 + 2s + 1}$$

Note that I did not substitute the initial conditions and $X(s)$ into the above equations. This shows clearly that the first term is the ZSR and the second ZIR. Now we can substitute these conditions back in and obtain

$$Y_{\text{ZSR}}(s) = \frac{1}{s^2 + 2s + 1}$$

$$Y_{\text{ZIR}}(s) = \frac{1}{s^2 + 2s + 1}$$

It turns that they are the same.

4. (3 pts) (Bounded and Transient signals) Without computing the inverse Laplace transform, determine which of the following signals are bounded and which are transient:

Solution:

(a) $X(s) = \dfrac{s-2}{s^2 + 7s + 12}$ is bounded and transient. $h(t)$ is a bounded signal because:

(a) Degree $(s-2) <$ Degree$(s^2 + 7s + 12)$

(b) The poles are at -3 and -4 (as $s^2 + 7s + 12 = (s+3)(s+4)$) and they are all on the open left half plane. $h(t)$ is transient because all the poles are on the open left half plane.

(b) $X(s) = \dfrac{s+3}{s^2 - 2s + 2}$ is not bounded and not transient. $h(t)$ is not a bounded signal because the poles are at $1+j$ and $1-j$ which are on the right half plane. For the same reason, $h(t)$ is not transient.

(c) $X(s) = \dfrac{s^2 - 1}{s^4 - 4s^3 + 8s^2 - 8s + 4}$ is not bounded and not transient. $h(t)$ is not a bounded signal because the poles are at $1+j$ (multiplicity=2) and $1-j$ (multiplicity=2) which are on the right half plane. For the same reason, $h(t)$ is not transient.

5. (6 points) (Zero-state and Zero-input responses) You are asked to investigate an unknown dynamic system. The only information you have is that it is governed by a first-order differential equation relating the input $x(t)$ and output $y(t)$:

$$\frac{d}{dt}y(t) + ay(t) = bx(t)$$

with unknown output initial condition $y(0) = c$.

(a) Write down the zero-input and zero-state response of the system based on the unknowns a, b and c.

Solution:

$$\begin{aligned} sY(s) - c + aY(s) &= bX(s) \\ (s+a)Y(S) &= bX(S) + c \\ Y(s) &= \frac{b}{s+a}X(s) + \frac{c}{s+a} \end{aligned} \tag{2}$$

Zero-input response:

$$\mathscr{L}^{-1}\left[\frac{c}{s+a}\right] = c\,e^{-at}u(t)$$

Zero-state response:

$$\mathscr{L}^{-1}\left[\frac{b}{s+a}X(s)\right] = b\,e^{-at}u(t) * x(t)$$

(b) In order to identify the system, you input the following signal $x(t) = e^{-t}u(t)$ and find that the output can be well-approximated by $y(t) = \left(\frac{19}{4}e^{-5t} + \frac{1}{4}e^{-t}\right)u(t)$. Find a, b and c.

Solution:

$$X(s) = \mathscr{L}\left[x(t)\right] = \frac{1}{s+1}$$

$$\begin{aligned} Y(s) &= \mathscr{L}\left[y(t)\right] \\ &= \frac{19}{4}\cdot\frac{1}{s+5} + \frac{1}{4}\cdot\frac{1}{s+1} \\ &= \frac{5s+6}{(s+5)(s+1)} \end{aligned} \tag{3}$$

We put $X(s) = \dfrac{1}{s+1}$ to Eq. 2 and get:

$$\begin{aligned} Y(s) &= \frac{b}{s+a}X(s) + \frac{c}{s+a} \\ &= \frac{b}{(s+a)}\cdot\frac{1}{(s+1)} + \frac{c}{(s+a)} \\ &= \frac{cs+b+c}{(s+a)(s+1)} \end{aligned} \tag{4}$$

Compare Eq. 3 and Eq. 4 we get: $a = 5, b = 1, c = 5$

(c) What is the steady state response of this system?

Solution: As the output $y(t) = \left(\frac{19}{4}e^{-5t} + \frac{1}{4}e^{-t}\right)u(t)$ decays to 0 as $t \to \infty$, the steady state response is 0.