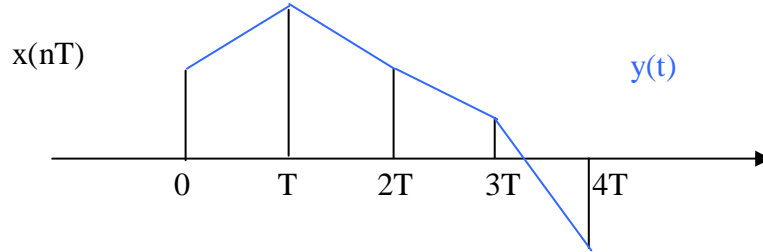


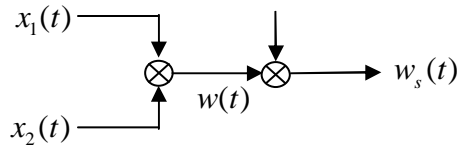
## EE422G Homework #9 Solution

1. (3 points) Sampling and Reconstruction

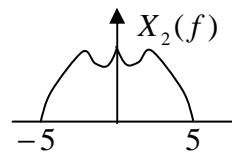
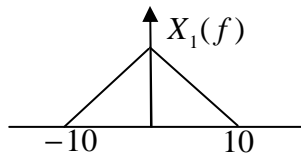
- a) Given the following discrete-time input  $x(nT)$ , sketch the outputs reconstructed continuous signal  $y(t)$  using Sample-And-Hold and Linear Interpolation.



- b) In the following system, two continuous-time functions  $x_1(t)$  and  $x_2(t)$  are MULTIPLIED and the product  $w(t)$  is sampled with sampling period  $T$ .



If the spectrums of  $x_1(t)$  and  $x_2(t)$  are given as follows, determine the maximum sampling period  $T$  such that  $w(t)$  is recoverable from  $w_s(t)$  through the use of an ideal lowpass filter.



Since multiplication in time domain is equivalent to convolution in frequency domain, the signal  $w(t) = x_1(t)x_2(t)$  is bandlimited within  $[-10-5 \text{ Hz}, 10+5 \text{ Hz}]$  or  $[-15 \text{ Hz}, 15\text{Hz}]$ . Thus the maximum sampling period  $T = 1/(15 \cdot 2) = 1/30$  seconds.

2. (2 points) An A/D converter has an input signal given by the Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Show the signal-to-noise ratio at the output of the A/D converter is given by

$$SNR = \frac{12}{D^2} 2^{2n} \sum_{n=-\infty}^{\infty} |X_n|^2$$

where  $D$  is the dynamic range.

Solution:

Signal-to-Noise Ratio (SNR) defined as:

$$SNR = \frac{\text{Average signal power}}{\text{Average power of quantization error}} \Big|_{in \text{ dB}}$$

From Parseval's theorem, the average signal power is  $\langle x^2(t) \rangle = \sum_{n=-\infty}^{\infty} |X_n|^2$ . The error  $e$  must be between  $-\Delta/2$  and  $\Delta/2$ , where  $\Delta$  is the interval between successive levels. Let's denote as  $D$ .

$$\Delta = \frac{D}{2^n} = D2^{-n}$$

Assume the quantization error  $e$  is uniformly distributed between  $-\Delta/2$  and  $\Delta/2$ ,

$$E[e^2] = \int_{-\Delta/2}^{\Delta/2} e^2 \frac{1}{\Delta} de = \frac{\Delta^2}{12} = \frac{D^2 2^{-2n}}{12}$$

$$SNR = \frac{|X_n|^2}{E[e^2]} = \frac{12}{D^2} 2^{2n} \sum_{n=-\infty}^{\infty} |X_n|^2$$

3. (4 points) An RC low-pass filter having transfer function

$$H(f) = \frac{1}{1 + j(f/f_3)}$$

where  $f_3$  is the 3-dB frequency of the filter, is used for reconstruction of a signal from its samples. We can ignore the multiplying constant  $T$  which can be realized as a multiplier external to the filter. The signal  $x(t)$  to be reconstructed is assumed to be a low-pass signal with bandwidth  $W$ . We wish to determine an appropriate sampling frequency so that the original spectrum  $X(f)$  is passed by the filter with negligible error and that the translated spectra  $X(f \pm f_s)$  were sufficiently attenuated.

- a. Let  $20 \log |H(W)| = \delta_1$  dB and  $20 \log |H(f_s - W)| = \delta_2$  dB. We clearly wish  $\delta_1$  to correspond to negligible attenuation relative to the dc gain and  $\delta_2$  to a large attenuation. For fixed  $\delta_1, \delta_2$  and signal bandwidth  $W$ , show that necessary sampling frequency,  $f_s$ , is

$$f_s = W \frac{\sqrt{10^{-0.1\delta_1} - 1} + \sqrt{10^{-0.1\delta_2} - 1}}{\sqrt{10^{-0.1\delta_1} - 1}}$$

- b. Use the result to determine the ratio  $f_s / W$  for
- $\delta_1 = -3$  dB and  $\delta_2 = -30$  dB
  - $\delta_1 = -1$  dB and  $\delta_2 = -30$  dB
  - $\delta_1 = -1$  dB and  $\delta_2 = -40$  dB

Comment on these results. How can large values of  $f_s / W$  be avoided?

Solution:

Since

$$H(f) = \frac{1}{1 + j \left(\frac{f}{f_3}\right)^2}$$

it follows that the amplitude response is

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_3}\right)^2}}$$

and the amplitude response, expressed in dB, is

$$20 \log_{10} |H(f)| = -10 \log_{10} \left( 1 + \left(\frac{f}{f_3}\right)^2 \right)$$

(a) Using the given definitions of  $\delta_1$ ,  $\delta_2$  gives

$$-10 \log_{10} \left( 1 + \left(\frac{W}{f_3}\right)^2 \right) = \delta_1$$

from which

$$\frac{W}{f_3} = \sqrt{10^{-0.1\delta_1} - 1}$$

and

$$\frac{f_s - W}{f_3} = \sqrt{10^{-0.1\delta_2} - 1}$$

since

$$\frac{f_s - W}{f_3} = \frac{f_s}{f_3} - \frac{W}{f_3} = \frac{f_s}{W} \frac{W}{f_3} - \frac{W}{f_3}$$

solve for  $f_s/W$  gives

$$\frac{f_s}{W} = \frac{\frac{W}{f_3} + \frac{f_s - W}{f_3}}{\frac{W}{f_3}}$$

substituting for  $\frac{f_s - W}{f_3}$  and  $\frac{W}{f_3}$  yields

$$f_s = W \frac{\sqrt{10^{-0.1\delta_1} - 1} + \sqrt{10^{-0.1\delta_2} - 1}}{\sqrt{10^{-0.1\delta_1} - 1}}$$

(b) i. for  $\delta_1 = -3dB$  and  $\delta_2 = -30dB$

$$\frac{f_s}{W} = \frac{\sqrt{10^{0.3} - 1} + \sqrt{10^3 - 1}}{10^{0.3} - 1} = 32.68$$

ii. for  $\delta_1 = -1dB$  and  $\delta_2 = -30dB$

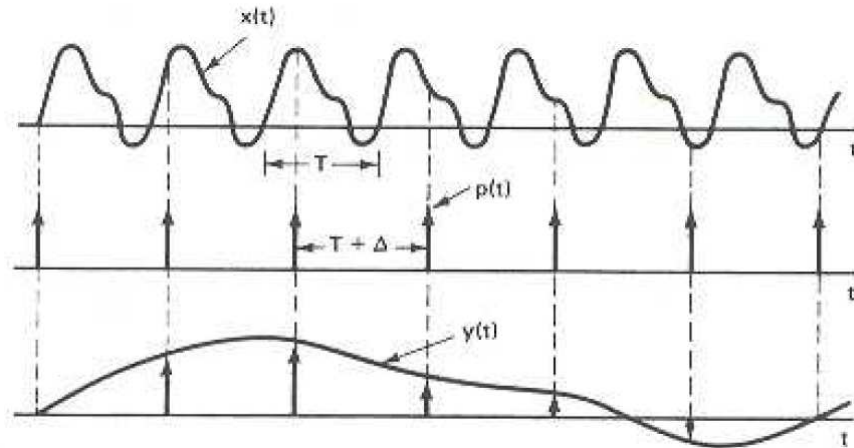
$$\frac{f_s}{W} = \frac{\sqrt{10^{0.1} - 1} + \sqrt{10^3 - 1}}{10^{0.1} - 1} = 123.07$$

iii. for  $\delta_1 = -1dB$  and  $\delta_2 = -40dB$

$$\frac{f_s}{W} = \frac{\sqrt{10^{0.1} - 1} + \sqrt{10^4 - 1}}{10^{0.1} - 1} = 196.51$$

Note that reducing the attenuation at  $f = W$  from -3 to -1 dB results in the necessity of increasing the sampling frequency by a factor of 4, while increasing the attenuation at  $f = f_s - W$  from -30 to -40 dB requires the sampling frequency to be increased by 60%. Large values of  $f_s/W$  can be avoided by using a higher-order filter.

4. (3 points) It is often necessary to display on an oscilloscope screen waveforms having very short time structures, for example, on the scale of thousandths of a nanosecond. Since the rise time of the fastest oscilloscope is longer than this, such display cannot be achieved directly. If however, the waveform is periodic, the desired result can be obtained indirectly using an instrument called a sampling oscilloscope. As shown in the figure below, the fast waveform  $x(t)$  is sampled once every period but at successively later points in successive periods. If the resulting impulse train is then passed through an appropriate low-pass interpolating filter the output,  $y(t)$ , will be proportional to the original fast waveform slowed down or stretched out in time; i.e.  $y(t)$  is proportional to  $x(at)$  where  $a < 1$ .



Assume that the input is a sinusoid  $x(t) = A + B\cos[(2\pi/T)t + \theta]$ . Draw a block diagram of a system that impulse train samples  $x(t)$  via  $p(t)$  and then “immediately” reconstructs  $y(t)$  from the impulse train sampled signal using a low-pass filter. What is the maximum cutoff of the low-pass filter? Find the range of  $\Delta$  so that  $y(t)$  is proportional to  $x(at)$  with  $a < 1$ .

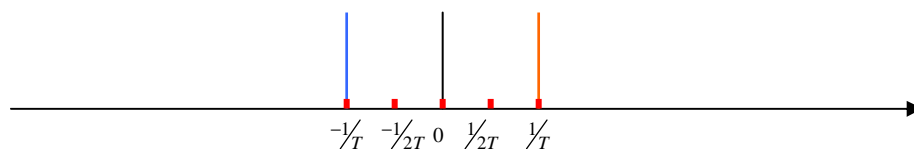
As described in the question, we wish to reconstruct  $y(t)$  based on the following block diagram:



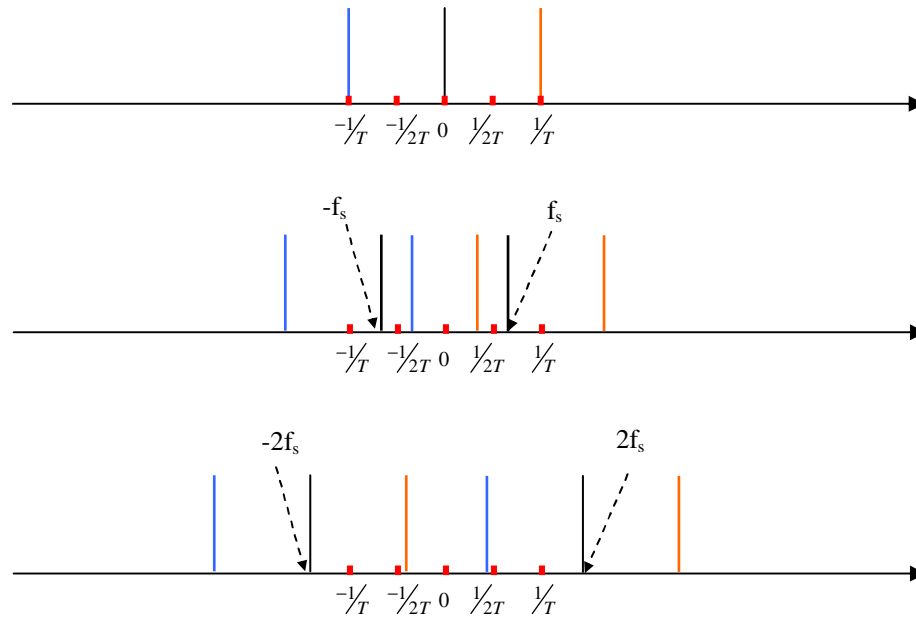
To analyze what the oscilloscope is doing, we start with the Fourier spectrum of the signal  $A + B\cos[(2\pi/T)t + \theta]$  which is

$$X(f) = A\delta(f) + \frac{Be^{j\theta}}{2}\delta(f - \frac{1}{T}) + \frac{Be^{-j\theta}}{2}\delta(f + \frac{1}{T})$$

Pictorially, the spectrum consists of three delta functions with different values:



Let  $f_s = 1/(T+\Delta) < 1/T$ . This is substantially less than the Nyquist rate which is  $2/T$ . The following shows the original spectrum and the first two periodic extensions to both the negative and positive directions. For clarity, they are shown in separate graphs even though they are all present simultaneously. We do not need to consider the third periodic extensions and beyond as their frequencies are too high to affect the low-pass interpolation filter.



It is clear from the above drawing that in order to have the low-pass region resembled the original spectrum, the red bar (second from the right) in the second graph must be closer to the origin than the blue bar (first from the right) in the third graph.

$$\text{Location of the red bar} = -f_s + 1/T = -1/(T+\Delta) + 1/T$$

$$\text{Location of the blue bar} = 2f_s - 1/T = 2/(T+\Delta) - 1/T$$

The condition is thus:  $-1/(T+\Delta) + 1/T < 2/(T+\Delta) - 1/T$

$$\Rightarrow 3/(T+\Delta) > 2/T$$

$$\Rightarrow \Delta < T/2$$

If this condition is satisfied, then the cutoff frequency can be set to the location of the rightmost position of the red bar in the second graph, i.e.  $-1/(T+\Delta) + 1/T = 1/(3T)$

5. (3 points) In order to earn extra travel money for your spring break vacation, you signed up for a temporary job collecting data at the Astronomy Department between 1:00am and 6:00am. The job was to take the reading  $x(nT)$  from a radio telescope every half an hour ( $T=30$ ) – thus you need to get 11 readings per night. As  $x(nT)$  does not change much from reading to reading, the job had become quite boring after the first two nights. On the third night, you had fallen asleep in your dorm room and did not get to the lab until 5:19 am. Hoping to somehow correct your mistakes, you frantically took down eleven readings everything 4 minutes starting at 5:20am.

On your way back to your dorm, you kept thinking how you could report the values to the Astronomy Professor. All of a sudden, you remembered about the sinc interpolation that you learned in EE422G:

$$x(t) = \sum_{n=0}^{\infty} x(nT) \operatorname{sinc}\left(\frac{t-nT}{T}\right)$$

The Astronomy professor already convinced you that  $T=30$  minutes is sufficient to produce a sampling rate higher than the Nyquist sampling frequency. Since your job were to take 11 readings, you conclude that you can reconstruct  $x(t)$  at any  $t$  within the five hour period ( $0 \leq t \leq 300$ ) using the following formula:

$$x(t) = \sum_{n=0}^{10} x(nT) \operatorname{sinc}\left(\frac{t-nT}{T}\right)$$

a. Now, you had the following 11 readings in your notebook:

x(260)	x(264)	x(268)	x(272)	x(276)	x(280)	x(284)	x(288)	x(292)	x(296)	x(300)
-0.266	-0.168	-0.058	0.060	0.180	0.297	0.403	0.491	0.553	0.586	0.588

Putting these values on the left hand side of the above equation, you formulate a system of 11 equations with 11 unknowns<sup>1</sup>  $x(0), x(T), x(2T), \dots, x(10T)$ . Use Matlab to find these eleven unknowns.

The following is the sample Matlab code to do this:

```
sampleValues = [-.266 -.168 -.058 .06 .18 .297 .403 .491 .553 .586 .588]';
sampleTime   = 260:4:300;
targetTime   = 0:30:300;
T            = 30;

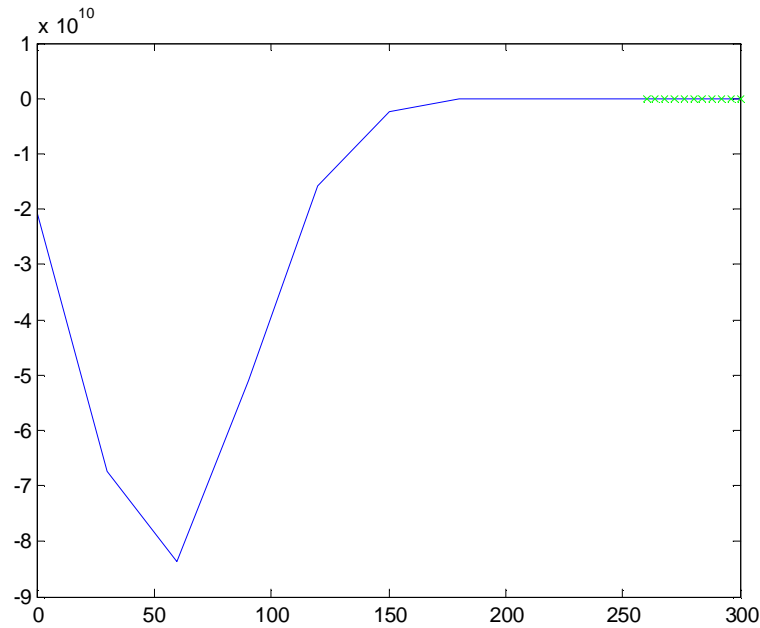
% Formulate the linear system of equation:
% A*targetValues = sampleValues
A = zeros(11,11);
for i=1:11,
    A(i,:) = sinc((sampleTime(i)-targetTime)/T);
end

% Given sampleValues, we solve for targetValues
% by left dividing (\) both sides by A. This is
% numerically more robust than multiplying inv(A).
targetValues = A\sampleValues;

plot(targetTime,targetValues,sampleTime,sampleValues,'gx');
```

The following is the resulting plot with green crosses indicating the observed samples.

<sup>1</sup> You actually know  $x(10T) = x(300)$  already but this is not important for this question.



- b. After giving yourself a few “high-fives”, you suddenly realize that something must be wrong – it means that the readings throughout five hours can be reconstructed using the readings captured within the last half an hour. In fact, it is not hard to see that they can be reconstructed using *any eleven readings captured within an arbitrary small period of time!* How can this be?

In part a), you have used readings within a 40-minute period to reconstruct all sample points within the entire five hour period. It is not hard to see that it is possible to infer sample points beyond this five hour period. As a result, you can see that the continuous-time signal is NOT ZERO before and after the five hour period – in fact, this signal  $x(t)$  goes on forever.

Signals in real-world, however, all have finite duration. The above analysis implies that real-world signals are not truly band-limited – i.e. the bandwidth of any finite-duration signals must be infinite and thus no finite-period sampling can fully reconstruct the signal. Nyquist sampling theorem only provides a mathematical model to approximate reality. It is intended to be used to **LOCALLY** reconstruct the continuous-time signal. Stretching its application to reconstruct the signal at distant future (or past) creates numerical instability and results in ridiculous conclusion such as the one depicted in this question.