Name: ______________________________________ Student ID: ________________

Midterm 3 (Fall 2006) of EE422G:

1. This midterm consists of eight single-sided pages. The first three pages contain various tables followed by FOUR exam questions and one extra worksheet. You can tear out any page but make sure all the pages you turn in have your name and student ID on them.
2. You have one hour to finish this exam.
3. You are allowed to use two double-sided page of cheat sheet.

Good luck!

	Properties of Z-transform		
1	Linearity	$a_1 x_1(nT) + a_2 x_2(nT)$	$a_1 X_1(z) + a_2 X_2(z)$
2	Multiply by a^n	$a^n x(nT)$	$X\left(\frac{z}{a}\right)$
3	Time Delay	$x(nT - mT)u(nT - mT), m > 0$	$z^{-m} X(z)$
4	Multiply by n	$nx(nT)$	$-z\frac{d}{dz}X(z)$
5	Initial Value Theorem	$x(0) = \lim_{z\to\infty} X(z)$	
6	Final Value Theorem	$x(\infty) = \lim_{z\to 1}(1 - z^{-1})X(z)$	
7	Time Convolution	$\sum_{m=0}^{\infty} x(mT)y(nT - mT)$	$X(z)Y(z)$

TABLE 8-1
Short Table of z-Transforms

Transform Pair Number	Continuous-time Function $f(t)$ for $t > 0$	Sample Values $f(nT)$ for $n \geq 0$	z-Transform of $f(nT)$
1.	—	$f(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \triangleq \delta(n)$	1
2.	1 (unit step)	1	$\frac{1}{1 - z^{-1}}$
3.	$e^{-\alpha t}$	$e^{-\alpha nT} = (e^{-\alpha T})^n = K^n$	$\frac{1}{1 - e^{-\alpha T}z^{-1}} = \frac{1}{1 - Kz^{-1}}$
4.	t	nT	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$
5.†	te^{-at}	nTe^{-anT}	$\frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$
6.†	$\sin bt$	$\sin bnT$	$\frac{(\sin bT)z^{-1}}{1 - 2(\cos bT)z^{-1} + z^{-2}}$
7.†	$\cos bt$	$\cos bnT$	$\frac{1 - (\cos bT)z^{-1}}{1 - 2(\cos bT)z^{-1} + z^{-2}}$
8.†	$e^{-at}\sin bt$	$e^{-anT}\sin bnT$	$\frac{e^{-aT}(\sin bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$
9.†	$e^{-at}\cos bt$	$e^{-anT}\cos bnT$	$\frac{1 - e^{-aT}(\cos bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$

†In transforms 5 through 9, e^{-aT} and bT can be replaced by constants, K_1 and K_2, respectively, as was done in transform 3. Convergence of the z-transform requires $|K_1| < 1$.

Name: ____ Student ID: ________________

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\displaystyle\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\displaystyle\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Name: ____ Student ID: ________________

	Properties	Time Domain	Laplace Transform
1	Linearity	$a_1x_1(t)+a_2x_2(t)+\ldots+a_nx_n(t)$	$a_1X_1(s)+a_2X_2(s)+\ldots+a_nX_n(s)$
2	Frequency Shifting	$e^{-\alpha t}x(t)$	$F(s+\alpha)$
3	Time Delay	$x(t-a)u(t-a)$	$e^{-\alpha s}X(s)$
4	Time Scaling	$x(\alpha t)$	$\frac{1}{\alpha}X\left(\frac{s}{\alpha}\right)$
5	Time Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
6	Time Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{X(s)}{s}+\frac{1}{s}\int_{-\infty}^{0^-}x(\tau)d\tau$
7	Initial Value Theorem	$\lim_{t\to 0^+}x(t)$	$\lim_{s\to\infty}sX(s)=x(0^+)$
8	Final Value Theorem	$\lim_{t\to\infty}x(t)$	$\lim_{s\to 0}sX(s)=x(\infty)$
9	Time Convolution	$x(t)*y(t)$	$X(s)Y(s)$

TABLE 5-3
Extended Table of Single-Sided Laplace Transforms

Signal	Laplace Transform	Comments on Derivation
1. $\delta^{(n)}(t)$	s^n	Direct evaluation with aid of (1-66)
2. 1 or $u(t)$	$\frac{1}{s}$	Direct evaluation
3. $\frac{t^n \exp(-\alpha t)u(t)}{n!}$	$\frac{1}{(s+\alpha)^{n+1}}$	Differentiation applied to pair 3, Table 5-1
4. $\cos\omega_0 t\, u(t)$	$\frac{s}{s^2+\omega_0^2}$	Example 5-1
5. $\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	Example 5-1
6. $\exp(-\alpha t)\cos\omega_0 t\, u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	s-shift and pair 4
7. $\exp(-\alpha t)\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	s-shift and pair 5

Name: ______________________________________ Student ID: ________________

1. (25 points) Discrete Fourier Transform and Convolution

   Let x(n) and y(n) be two three-point sequence:

$$x(n) = \begin{cases} 1 & \text{for n} = 0 \\ 2 & \text{for n} = 1 \\ 1 & \text{for n} = 2 \end{cases} \qquad y(n) = \begin{cases} -1 & \text{for n} = 0 \\ 2 & \text{for n} = 1 \\ 1 & \text{for n} = 2 \end{cases}$$

   a. (9 points) Compute the 3-point *circular convolution* z(n) between x(n) and y(n).

$$z(0) = 1\cdot(-1) + 2\cdot 1 + 1\cdot 2 = 3$$
$$z(1) = 1\cdot 2 + 2\cdot(-1) + 1\cdot 1 = 1$$
$$z(2) = 1\cdot 1 + 2\cdot 2 + 1\cdot(-1) = 4$$

   b. (10 points) Compute the *5-point* DFT X(k) for x(n). You do not need to simplify your answers.

$$X(0) = 1 + 2 + 1 = 4$$
$$X(1) = 1 + 2\cdot\exp\left(-j\frac{2\pi}{5}\right) + \exp\left(-j\frac{4\pi}{5}\right)$$
$$X(2) = 1 + 2\cdot\exp\left(-j\frac{4\pi}{5}\right) + \exp\left(-j\frac{8\pi}{5}\right)$$
$$X(3) = 1 + 2\cdot\exp\left(-j\frac{6\pi}{5}\right) + \exp\left(-j\frac{12\pi}{5}\right)$$
$$X(4) = 1 + 2\cdot\exp\left(-j\frac{8\pi}{5}\right) + \exp\left(-j\frac{16\pi}{5}\right)$$

   c. (6 points) In addition to X(k), suppose you have also computed the 5-point DFT Y(k) for y(n). Without carrying out any computations, comment on whether applying a 5-point DFT on z(n) (part a) will result in the product of X(k) and Y(k). Explain your answer.

   No. When multiplying X(k) with Y(k), one obtains the 5-point DFT of the linear convolution between x(n) and y(n), not the 3-point circular convolution z(n).

Name: ______________________________________ Student ID: ________________

2. (25 points) Fast Fourier Transform

a. (5 points) Name one reason why FFT is very commonly used in signal processing applications.

FFT can be executed very efficiently.

b. (5 points) We discuss two different FFT implementations in lecture and in homework. Name these two implementations.

Decimation-in-time and Decimation-in-frequency.

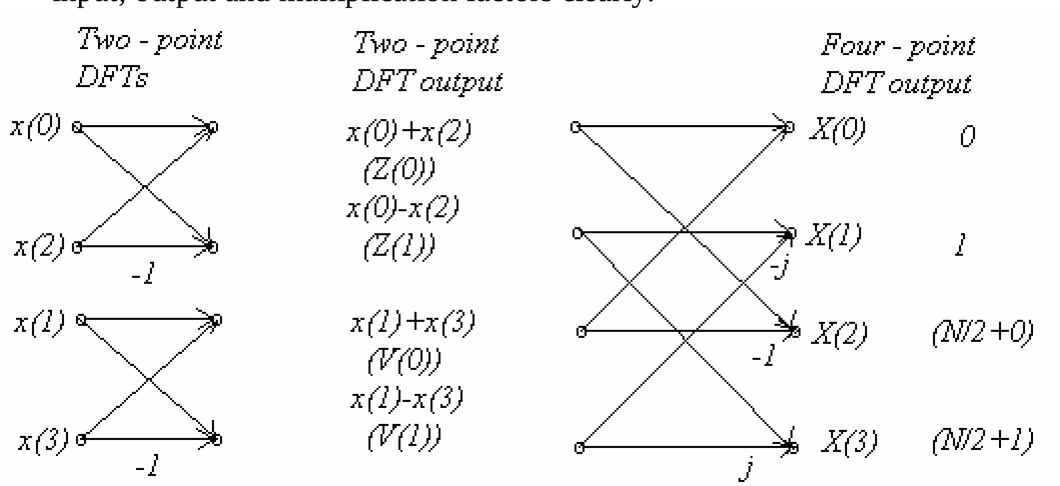
c. (5 points) What is the relationship between DFT and DTFT?

DFT (Discrete Fourier Transform) are frequency samples of DTFT (Discrete-time Fourier Transform)

d. (5 points) Complete the missing entries in the following table.

Signals	**Fourier**	**Transform Characteristics**
Continuous in t & Periodic	Fourier Series	Discrete in ω
Continuous in t	Continuous-time Fourier Transform	Continuous in ω
Discrete in t	Discrete-time Fourier Transform	Continuous in ω & Periodic
Discrete in t & Periodic	Discrete Fourier Transform	Discrete in ω & Periodic

e. (5 points) Show the internal structure of a 4-point FFT. Make sure you label the input, output and multiplication factors clearly.



Name: ______________________________________ Student ID: ________________

3. (25 points) A linear system is described by the following transfer function:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-4 & 1\\ -2 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}0\\ 2\end{pmatrix}u$$

$$y=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

a) (8 points) Write down the A, B, C, D matrices in the standard form corresponding to the above equations.

$$A=\begin{pmatrix}-4 & 1\\ -2 & -1\end{pmatrix}, B=\begin{pmatrix}0\\ 2\end{pmatrix}, C=\begin{pmatrix}1 & 0\end{pmatrix}, D=0$$

b) (7 points) Write down the transfer function H(s).

$$A=\begin{pmatrix}-4 & 1\\ -2 & -1\end{pmatrix} \quad B=\begin{pmatrix}0\\ 2\end{pmatrix} \quad C=\begin{pmatrix}1 & 0\end{pmatrix} \quad D=0$$

$$H(s)=C(sI-A)^{-1}B+D$$

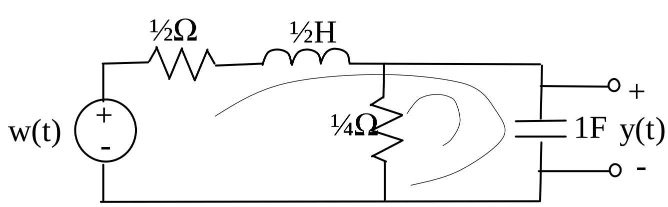
$$=\begin{pmatrix}1 & 0\end{pmatrix}\left[\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix}-\begin{pmatrix}-4 & 1\\ -2 & -1\end{pmatrix}\right]^{-1}\begin{pmatrix}0\\ 2\end{pmatrix}+0$$

$$=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}s+4 & -1\\ 2 & s+1\end{pmatrix}^{-1}\begin{pmatrix}0\\ 2\end{pmatrix}$$

$$=\frac{1}{(s+4)(s+1)+2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}s+1 & 1\\ -2 & s+4\end{pmatrix}\begin{pmatrix}0\\ 2\end{pmatrix}$$

$$=\frac{2}{s^2+5s+6}$$

c) (10 points) If the same set of state-variable equations also represents the following circuit, which physical quantities of the circuit do the state variables x_1, x_2 represent? You must justify your answers.



Based on the output equation, we can easily deduce that x_1 is the voltage across the 1F capacitor. By setting x_2 as the inductor current, we can write

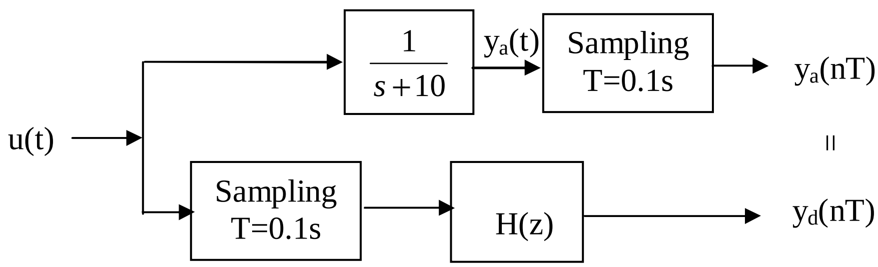
KVL around the bigger loop: $\dot{x}_2=-2x_1-x_2+2w$

KVL around the smaller loop: $\dot{x}_1=-4x_1+x_2$

which are the same as the state equations.

Name: ______________________________ Student ID: ______________

4. (25 points) An alternative to the impulse-invariant technique is *step-invariance* synthesis. The step-invariant filter is derived by placing a unit step on the input of an analog filter and a sampled unit step on the input to a digital filter. The digital filter H(z) is computed so that the output of the digital filter represents samples of the output of the analog filter.



a) (5 points) Write down the Laplace transform of $y_a(t)$.

Since the Laplace transform of the step function is 1/s, we have

$$L[y_a(t)] = \frac{1}{s(s+10)}$$

b) (10 points) Compute the time-domain output $y_a(nT)$ of the analog filter and then its Z-transform $Y_a(z)$.

$$L[y_a(t)] = \frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10} = \frac{1/10}{s} - \frac{1/10}{s+10} \Rightarrow y_a(t) = \frac{1}{10}u(t) - \frac{1}{10}e^{-10t}u(t)$$

$$y_a(nT) = \frac{1}{10}u(nT) - \frac{1}{10}e^{-n}u(nT)$$

$$Y_a(z) = \frac{1/10}{1-z^{-1}} - \frac{1/10}{1-e^{-1}z^{-1}}$$

c) (5 points) Write down the Z transform of the output $Y_d(z)$ in terms of H(z)

$$Y_d(z) = \frac{H(z)}{1-z^{-1}}$$

d) (5 points) By equating $Y_d(z)$ and $Y_a(z)$, compute H(z)

$$H(z) = \frac{1}{10} - \frac{1/10(1-z^{-1})}{1-e^{-1}z^{-1}}$$

Name: ____ Student ID: _________________

EXTRA WORKSHEET