8.3A Z-Transform

Importance:
Z-transform to discrete-time ⇔ Laplace transform to continuous-time

Definition:
Given signal \( x(nT) \) for \( n = 0,1,2,\ldots, \)
\[
Z(x(nT)) = X(z) = \sum_{n=0}^{\Delta} x(nT)z^{-n} = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots
\]

Motivation:
Let’s say \( x(nT) = \left(\frac{1}{2}\right)^{nT} u(nT) \)

P.S. \( u(nT) \) is the discrete-time step sequence.

Recall the continuous-time surrogate
\[
x_s(t) = \sum_{nT=-\infty}^{\infty} x(nT)\delta(t - nT) = \sum_{nT=0}^{\infty} \left(\frac{1}{2}\right)^{nT} \delta(t - nT)
\]

Taking its Laplace transform
\[
L[x_s(t)] = \sum_{nT=0}^{\infty} \left(\frac{1}{2}\right)^{nT} L[\delta(t - nT)]
\]
\[
= \sum_{nT=0}^{\infty} \left(\frac{1}{2}\right)^{nT} e^{-nTs}
\]
\[
= \frac{1}{1 - \left(\frac{1}{2}\right)^{t} e^{-Ts}}
\]

Note that the final expression is NOT a polynomial in \( s \) – not very convenient to use in practice. We use the following substitution:
\[
z = e^{Ts}
\]
The final expression will become \[
\frac{1}{1 - \left(\frac{1}{2}\right)^{t} z^{-1}} = \frac{z}{z - \left(\frac{1}{2}\right)^{t}} .
\]

In general,
\[
L[x_s(t)] = \sum_{nT=0}^{\infty} x(nT)e^{-nTs} \overset{z=e^{Ts}}{=} \sum_{nT=0}^{\infty} x(nT)z^{-n} = Z[x(nT)]
\]
Mapping \( z = e^{\sigma T}, T > 0 \)

We are interested in mapping the ROC in s-plane to the z-plane – so that we can investigate the stability and other properties directly from the z-plane.

Let \( \sigma = \text{Re}(s) \) and \( \omega = \text{Im}(s) \),

\[
z = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T}
\]

\[
\Rightarrow |z| = e^{\sigma T}, \angle z = \omega T
\]

The real part \( \sigma = \text{Re}(s) \) only affects the modulus of \( z \). As the ROC in s-plane is based only on \( \sigma = \text{Re}(s) \), let’s see how it maps to z-plane:

- \( \sigma = -\infty \) → \( |z| = e^{-\infty T} = 0 \)
- \( \sigma = 0 \) → \( |z| = e^{0T} = 1 \)
- \( \sigma = +\infty \) → \( |z| = e^{+\infty T} = \infty \)

In words:
- Negative infinity in s-plane → Origin in z-plane
- Imaginary axis in s-plane → Unit circle in z-plane
- Positive infinity in s-plane → Positive infinity in z-plane

Pictorially:
Consequences:

1. Left half s-plane $\rightarrow$ Inside the unit circle in z-plane
   Right half s-plane $\rightarrow$ Outside of the unit circle in z-plane
   Imaginary axis in s-plane $\rightarrow$ Unit circle in z-plane
2. ROC in S-plane $\{s: \text{Re}(s) > p\} \rightarrow$ ROC in Z-plane $\{z: |z| > r\}$

3. $H(s)$ is BIBO stable if all poles are on the left half plane $\rightarrow$ $H(z)$ is BIBO stable if all poles are inside the unit circle (more later)
4. Fourier Transform in S-plane is evaluated along $j\omega$-axis $\rightarrow$ Fourier Transform in Z-plane is evaluated along the unit circle
   \[ X_s(e^{j\omega T}) \text{ must be the same as } X_r(e^{j(\omega T + 2n\pi)}) \text{, i.e. integral number of full rotation. As } j(\omega T + 2n\pi) = j(\omega + \frac{2n\pi}{T}) T \text{, the Fourier transform is a periodic function in } \omega \text{ with period } = \frac{2\pi}{T} = \omega_s. \text{ We have seen this before!} \]

\[ X_s(j\omega) \downarrow \text{after sampling} \]

\[ X_r(j\omega) \]
Examples of Z-transforms

Example 8-4: $z$-transform of unit pulse $\delta(n)$:

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \triangleq \delta(n)$$

$$X(z) = 1 + 0 \cdot z^{-1} + \cdots = 1$$

Example 8-5  $z$-Transform of unit step sequence $u(nT)$:

$$x(nT) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$X(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}}$$

with ROC = $\{z: |z| > 1\}$

Typically, we write the $Z$-transform in terms of $z^{-1}$ rather than $z$ as casual sequences contain only the negative powers of $z$.

Poles and Zeros in Z-transform:

$$X(z) = \frac{1}{1 - cz^{-1}}$$ has a pole at $z = c$ and a zero at $z = 0$

$$X(z) = \frac{1 - cz^{-1}}{z^{-1}}$$ has a zero at $z = c$ and a pole at $z = \infty$

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<td>$\sum_{m=0}^{\infty} x(mT)y(nT - mT)$</td>
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Multiply by $a^n$

$$Z[a^n x(nT)] = \sum_{n=0}^{\infty} a^n x(nT) z^{-n} = \sum_{n=0}^{\infty} x(nT) \left( \frac{z}{a} \right)^n = X \left( \frac{z}{a} \right)$$

Ex: Find the Z-transform of $y(nT) = e^{-aT} u(nT)$

Rewrite $y(nT) = (e^{-aT})^n u(nT)$. Given $Z[u(nT)] = \frac{1}{1-z^{-1}}$,

$$Z[y(nT)] = \frac{1}{1 - (\frac{z}{e^{-aT}})^{-1}} = \frac{1}{1 - e^{-aT} z^{-1}}$$

Multiply by $n$

$$-z \frac{dX(z)}{dz} = -z \frac{d}{dz} \left( x(0) + x(T) z^{-1} + x(2T) z^{-2} + x(3T) z^{-3} + \ldots \right)$$

$$= -z \left( -x(T) z^{-2} - 2x(2T) z^{-3} - 3x(3T) z^{-4} - \ldots \right)$$

$$= 0 \cdot x(0) + 1 \cdot x(T) z^{-1} + 2 \cdot x(2T) z^{-2} + 3 \cdot x(3T) z^{-3} + \ldots = Z(nx(nT))$$

Ex. Find the Z-transform of $y(nT) = nT u(nT)$

Since $Z[T u(nT)] = \frac{T}{1-z^{-1}}$, $Z[nT u(nT)] = -z \frac{d}{dz} \frac{T}{1-z^{-1}} = \frac{T z^{-1}}{(1-z^{-1})^2}$

Initial Value Theorem

$$X(z) = x(0) + x(1) z^{-1} + \ldots \text{. Having } z \text{ goes to } \infty \text{ kills every term except the first.}$$

Final Value Theorem

Final value $x(\infty) = \lim_{z \to 1} (1-z^{-1}) X(z)$ if $X(z)$ does not have any poles on or outside the unit circle, except possibly a simple pole at $z=1$.

A formal proof is beyond the scope but here is an informal proof:

$$\begin{align*}
(1-z^{-1}) X(z) &= x(0) + [x(T) - x(0)] z^{-1} + [x(2T) - x(T)] z^{-2} + [x(3T) - x(2T)] z^{-3} + \ldots \\
&= x(0) + x(T) - x(0) + x(2T) - x(T) + x(3T) - x(2T) = x(\infty)
\end{align*}$$
### 8.3B Inverse Z-Transform

Two Basic Methods:
1. Express $X(z)$ into "Definition Form"  
   $X(z) = x(0) + x(1)z^{-1} + \ldots$

   ex. $X(z) = 1 + 2z^{-1} - 5z^{-2} - 2z^{-3} + z^{-4}$

   This implies $x(0) = 1, x(T) = 2, x(2T) = -5, x(3T) = -2, x(4T) = 1$, and zero otherwise.

   ex. Long division $X(z) = \frac{1}{(1-az^{-1})}$

   $1 + az^{-1} + a^2 z^{-2} + ...$

   $1 - az^{-1}$

   $\Rightarrow x(nT) = a^n u(nT)$
2. Express $X(z)$ into partial-fraction from

Expand $X(z) = \sum_{k=0}^{N} \alpha_k z^{-k} + \sum_{k=0}^{N} \frac{\beta_k}{(1 - a_k z^{-1})^m}$, all in terms of $z^{-1}$

↑

↑

each term has an inverse transform

Important: before doing partial-fraction expansion, make sure the $z$-

transform is in proper rational function of $z^{-1}$ !

Example 8.9

$$X(z) = \frac{z^2}{(z-1)(z-0.2)} = \frac{1}{(1-z^{-1})(1-0.2z^{-1})}$$

Solution : $X(z) = \frac{1}{(1-z^{-1})(1-0.2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$

Heaviside’s Expansion Method:

(1) $(1-z^{-1})X(z) = \frac{1}{1-0.2z^{-1}} = A + \frac{B(1-z^{-1})}{1-0.2z^{-1}} \Rightarrow$

$$\frac{1}{1-0.2} = A + \frac{B \cdot 0}{1-0.2} \Rightarrow A = \frac{1}{0.8} = 1.25$$

(2) $(1-0.2z^{-1})X(z) = \frac{A(1-0.2z^{-1})}{1-z^{-1}} + \frac{B(1-0.2z^{-1})}{1-0.2z^{-1}} \Rightarrow$

$$\frac{1}{1-z^{-1}} = \frac{A(1-0.2z^{-1})}{1-z^{-1}} + B \frac{1-0.2z^{-1} = 0 (z=0.2)} \Rightarrow B = -0.25$$

$$X(z) = \frac{1.25}{1-z^{-1}} + \frac{-0.25}{1-0.2z^{-1}} \Rightarrow x(nT) = 1.25 - 0.25(0.2)^n$$