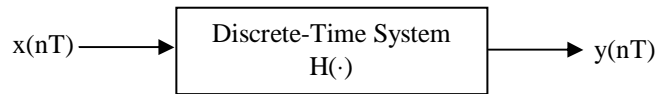


8-4A Difference Equation and Discrete-Time Systems



Definitions of various discrete-time properties:

Properties	Definitions
Linearity	$H[a \cdot x_1(nT) + b \cdot x_2(nT)] = a \cdot H[x_1(nT)] + b \cdot H[x_2(nT)]$
Shift-Invariant	$H[x(nT - kT)] = y(nT - kT)$
Casual	$y(nT)$ depends on $x(kT)$ for $k \leq n$
BIBO Stable	$x(nT)$ is bounded $\Rightarrow y(nT)$ is bounded

These are not new concepts – you have seen them in continuous-time system. In fact, one can treat discrete-time system as continuous-time system by using the continuous-time surrogates $x_s(t)$ and $y_s(t)$.

On the other hands, a strong motivation to use discrete-time system is its ease of implementation using digital logic. Thus, we want to do all things “discrete”.

In this section, we want to show:

1. The output of all Linear Shift-Invariant (LSI) discrete-time system can be computed by convolution with its IMPULSE RESPONSE.
2. Determine if a LSI system is causal and/or BIBO stable based on its impulse response.
3. Alternative representation of LSI system: difference equation
4. Classification and implementation of impulse responses: FIR and IIR

LSI System

Similar to the continuous-time relationship $x(t) = \int x(\tau)\delta(t-\tau)d\tau$, any discrete signal $x(nT)$ can be written as

$$x(nT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(nT - kT)$$

Feed $x(nT)$ to a LSI system H :

$$\begin{aligned} y(nT) &= H[x(nT)] \\ &= H\left[\sum_{k=-\infty}^{\infty} x(kT)\delta(nT - kT)\right] \\ &= \sum_{k=-\infty}^{\infty} x(kT)H[\delta(nT - kT)] \quad \text{Linearity} \end{aligned}$$

Define the impulse response $h(nT) = H[\delta(nT)]$.

As H is shift invariant, we have

$$h(nT - kT) = H[\delta(nT - kT)]$$

Thus,

$$\begin{aligned} y(nT) &= \sum_{k=-\infty}^{\infty} x(kT)h(nT - kT) \\ &\hat{=} x(nT) * h(nT) \end{aligned}$$

The output of any discrete-time LSI system can be written as the convolution between the input $x(nT)$ and the impulse response $h(nT)$.

We can actually simplify the convolution expression a bit.

First, assume $x(kT) = 0$ for $k < 0$, we have

$$y(nT) = \sum_{k=0}^{\infty} x(kT)h(nT - kT)$$

What if the system is also causal?

If so, $y(nT)$ can only depend on $x(kT)$ for $k \leq 0$. This implies that

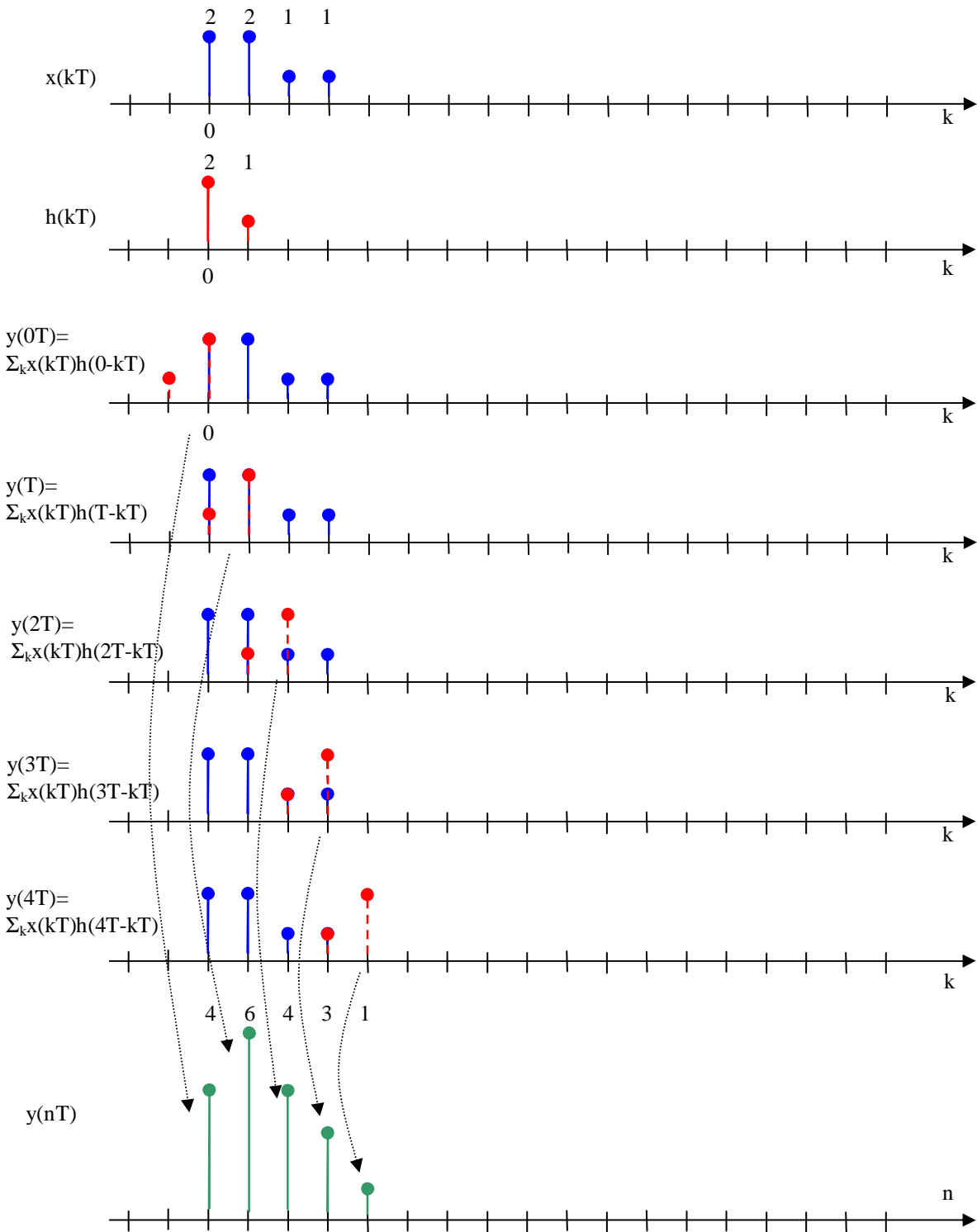
$$\begin{aligned} h(nT - kT) &= 0 \quad \text{for } k < n \\ \Leftrightarrow h(nT) &= 0 \quad \text{for } n < 0 \end{aligned}$$

For a casual LSI system,

$$y(nT) = \sum_{k=0}^n x(kT)h(nT - kT)$$

Notice that the summation reduces from infinitely many terms to only $n+1$ terms.

Let's do one convolution by hand



Note that the duration of the output $y(nT)$ is longer than that of $x(nT)$. In general, if $x(nT)$ has N samples and $h(nT)$ has K samples, $y(nT)$ has $N+K-1$ samples.

Z-transform in action

Time convolution is easy for finite duration impulse response. What if the impulse response is infinitely long?

Answer: Use Z-transform

We want to show for any causal LSI system:

$$y(nT) = \sum_{k=0}^{\infty} x(kT)h(nT - kT) \Leftrightarrow Y(z) = H(z)X(z)$$

$$\begin{aligned} Z(y(nT)) &= Z\left\{\sum_{k=0}^{\infty} x(kT)h(nT - kT)\right\} \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} x(kT)h(nT - kT) \right] z^{-n} \\ &= \sum_{k=0}^{\infty} \left[x(kT) \sum_{n=0}^{\infty} h(nT - kT) z^{-n} \right] \\ &= \sum_{k=0}^{\infty} \left[x(kT) \sum_{l=-k}^{\infty} h(lT) z^{-l-k} \right] \\ &= \sum_{k=0}^{\infty} \left[x(kT) z^{-k} \sum_{l=-k}^{\infty} h(lT) z^{-l} \right] \\ &= \sum_{k=0}^{\infty} \left[x(kT) z^{-k} H(z) \right] \\ &= H(z)X(z) \end{aligned}$$

Again, we call $H(z)$ the TRANSFER FUNCTION of the discrete-time system.

Example:

$$x(nT) = \left(\frac{1}{4}\right)^n u(nT)$$

$$h(nT) = \left(\frac{1}{3}\right)^n u(nT)$$

Find $x(nT)*h(nT)$

Solution:

First, compute the Z-transform $X(z)$ and $H(z)$

$$X(z) = Z\left[\left(\frac{1}{4}\right)^n u(nT)\right] = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$H(z) = Z\left[\left(\frac{1}{3}\right)^n u(nT)\right] = \frac{1}{1 - \frac{1}{3}z^{-1}}$$

Then compute their product:

$$Y(z) = X(z)H(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

Apply partial fraction :

$$Y(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{-3}{1 - \frac{1}{4}z^{-1}} + \frac{4}{1 - \frac{1}{3}z^{-1}}$$

Then,

$$y(nT) = Z^{-1}(Y(z)) = \left(-3\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{3}\right)^n\right)u(nT)$$

DONE!

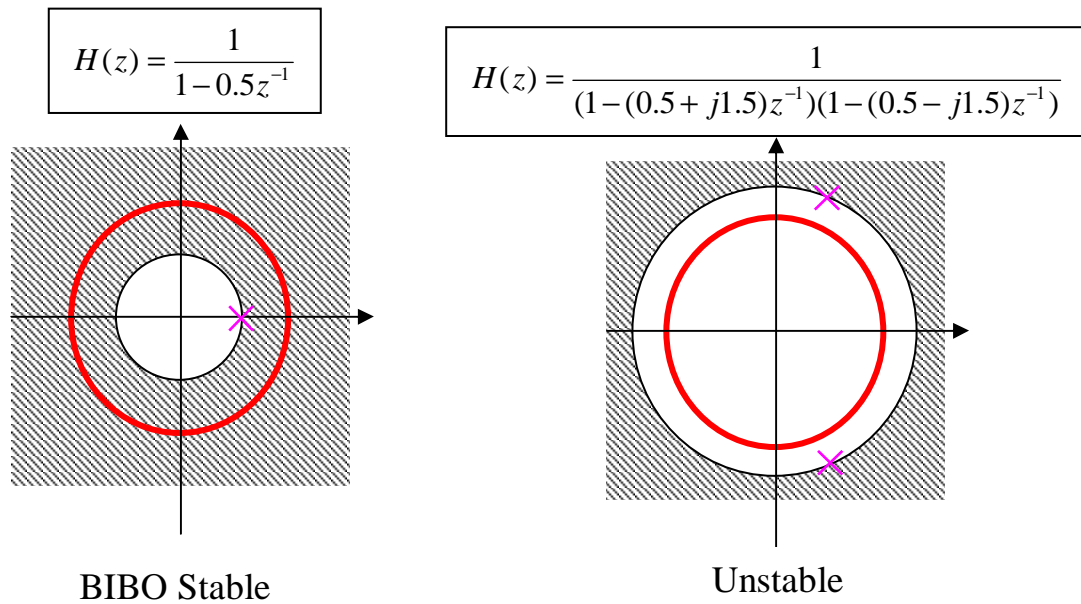
Just like Laplace transform, we can also determine whether a system is BIBO stable based on the pole locations of the transfer function.

Recall $H(s)$ is BIBO stable if its ROC region includes the imaginary axis.

And the imaginary axis in the s-plane maps to the unit circle in z-plane.

Thus, we have

$H(z)$ is BIBO stable if and only if its ROC contains the unit-circle in the z-plane.

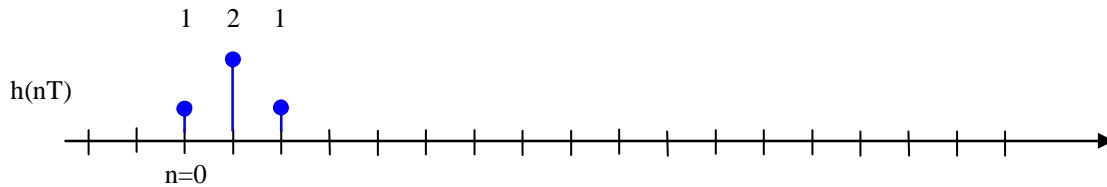


Types of Impulse Responses & Difference Equation

Two main types of impulse responses:

I. Finite Impulse Response (FIR)

- $h(nT)$ has finitely many non-zero values. Example



$$H(z) = 1 + 2z^{-1} + z^{-2} = (1 + z^{-1})^2$$

- Easy to see that if $H(z)$ is FIR, there is no denominator so it does not have any finite poles \Rightarrow the corresponding system is always BIBO stable

Express it in terms of the input $x(nT)$ and output $y(nT)$. Recall

$$\frac{Y(z)}{X(z)} = H(z) = 1 + 2z^{-1} + z^{-2}$$

$$Y(z) = X(z) + 2z^{-1}X(z) + z^{-2}X(z)$$

Taking the inverse Z-transform, we have

$$y(nT) = x(nT) + 2x(nT - T) + x(nT - 2T) \tag{1}$$

This is similar to the approach of recovering the differential equation from the transfer function in continuous-time. In discrete-time, (1) is called the **Difference Equation**. In general, the difference equation of any FIR system can be expressed as follows:

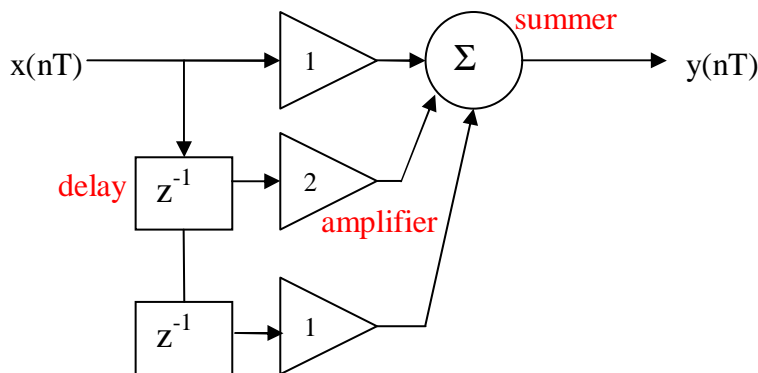
$$y(nT) = \sum_{k=0}^N a_k x(nT - kT)$$

Why difference equation?

Great for implementation – the followings are from Section 9-1, 9-2.

Example: $y(nT) = x(nT) + 2x(nT - T) + x(nT - 2T)$

Using simple delay, amplifier, and summer, we can implement this difference equation as:



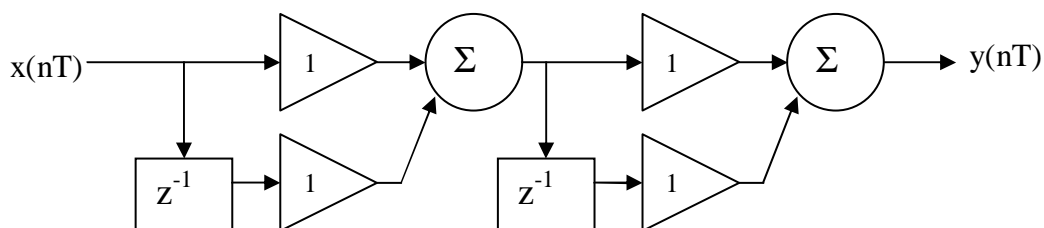
This structure is called the **Direct Form**.

However, this is not the only type of implementation. Notice that the corresponding Z-transform of the difference equation is

$$H(z) = 1 + 2z^{-1} + z^{-2} = (1 + z^{-1})^2$$

Thus, the system can also be implemented as applying the following difference equation twice:

$$y'(nT) = x'(nT) + x'(nT - T)$$



This is called the **Cascade Form**. Since any polynomial can be factored into linear and quadratic factors, the Cascade Form can be thought of as connecting basic linear and quadratic Direct Forms in series.

Cascade versus Direct Form:

1. Direct form uses fewer components.
2. Cascade form has the same basic building blocks.
3. Quantization in gain coefficients affects all zeros in Direct Form, but only individual components in Cascade form.

II. Infinite Impulse Response (IIR)

- $h(nT)$ has finitely many non-zero values.

Example: $h(nT) = (-15(\frac{1}{4})^n + 16(\frac{1}{3})^n)u(nT) \Leftrightarrow H(z) = \frac{1+z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})}$

- Unlike FIR, IIR have poles \Rightarrow A IIR may not be BIBO stable unless all its poles are within the unit circle.
- Difference Equation:

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1+z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}}$$

$$Y(z) - \frac{7}{12}z^{-1}Y(z) + \frac{1}{12}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

Taking the Inverse Z-transform:

$$y(nT) - \frac{7}{12}y(nT-T) + \frac{1}{12}y(nT-2T) = x(nT) + x(nT-T)$$

$$y(nT) = x(nT) + x(nT-T) + \frac{7}{12}y(nT-T) - \frac{1}{12}y(nT-2T)$$

In general, the difference equation for any IIR system can be expressed as

$$y(nT) = \sum_{k=0}^N a_k x(nT-kT) + \sum_{l=1}^M b_l y(nT-lT)$$

Compare this with the FIR equation: $y(nT) = \sum_{k=0}^N a_k x(nT-kT)$

FIR depends **only on the INPUT** values $x(nT)$. IIR depends on both **the input** $x(nT)$ and **the past OUTPUT** values $y(nT-T)$, $y(nT-2T)$, ... Thus, IIR contains feedback.

IIR has more varieties of implementation options than FIR. Just like FIR, the different implementations are due to different way of writing the Z-transform.

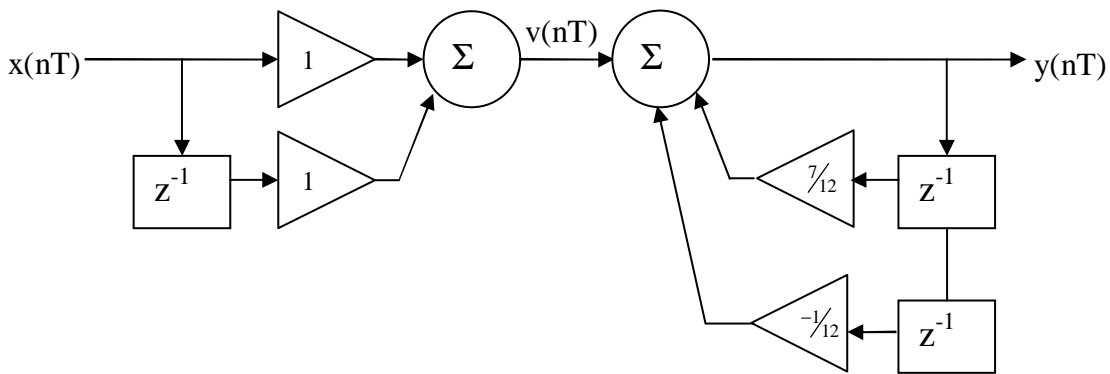
Example: $H(z) = \frac{1+z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})}$

Four different ways to write H(z)

1. $H(z) = (1+z^{-1}) \cdot \frac{1}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}} \Rightarrow \text{FIR} \rightarrow \text{Feedback} = \text{Direct Form I}$

FIR : $v(nT) = x(nT) + x(nT-T)$

Feedback : $y(nT) = v(nT) + (7/12)y(nT-T) - (1/12)y(nT-2T)$

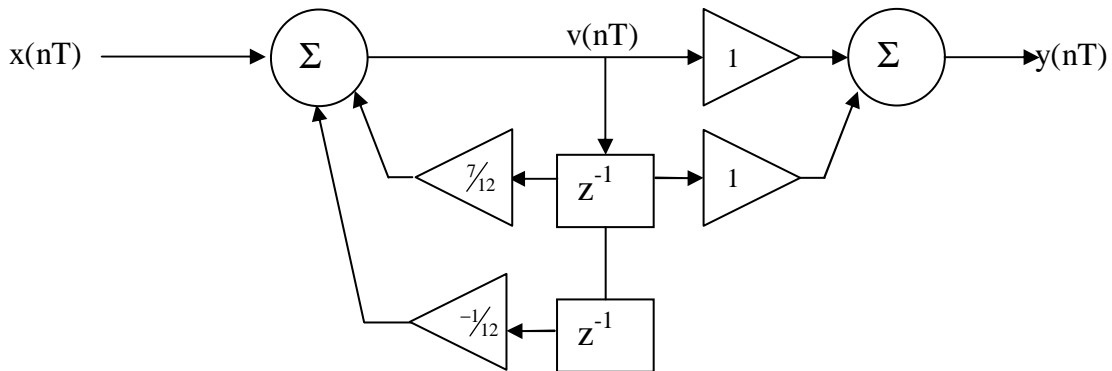


Remember the sign change in the feedback gains as compared with those from the transfer function.

2. $H(z) = \frac{1}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}} \cdot (1+z^{-1}) \Rightarrow \text{Feedback} \rightarrow \text{FIR} = \text{Direct Form II}$

Feedback : $v(nT) = x(nT) + (7/12)v(nT-T) - (1/12)v(nT-2T)$

FIR : $y(nT) = v(nT) + v(nT-T)$

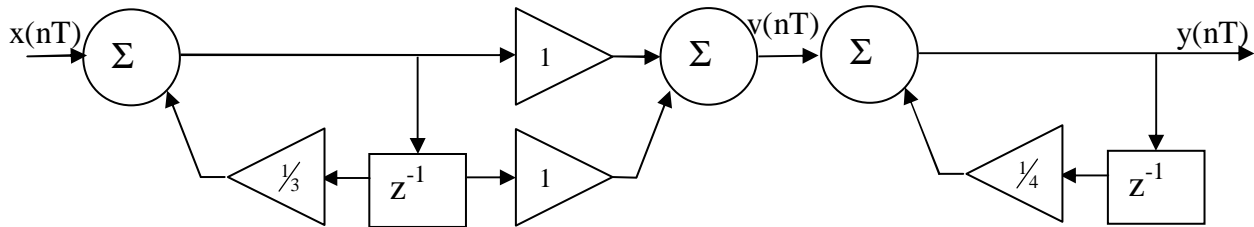


Direct Form II share delay elements between the Feedback and the FIR. (see highlighted terms), thus using fewer delay elements than Direct Form I.

3. $H(z) = \frac{1+z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})}$ Factorized into series of direct form II \Rightarrow cascade

First block: $v(nT) = x(nT) + x(nT-T) + (1/3)v(nT-T)$

Second block: $y(nT) = v(nT) + (1/4)y(nT-T)$

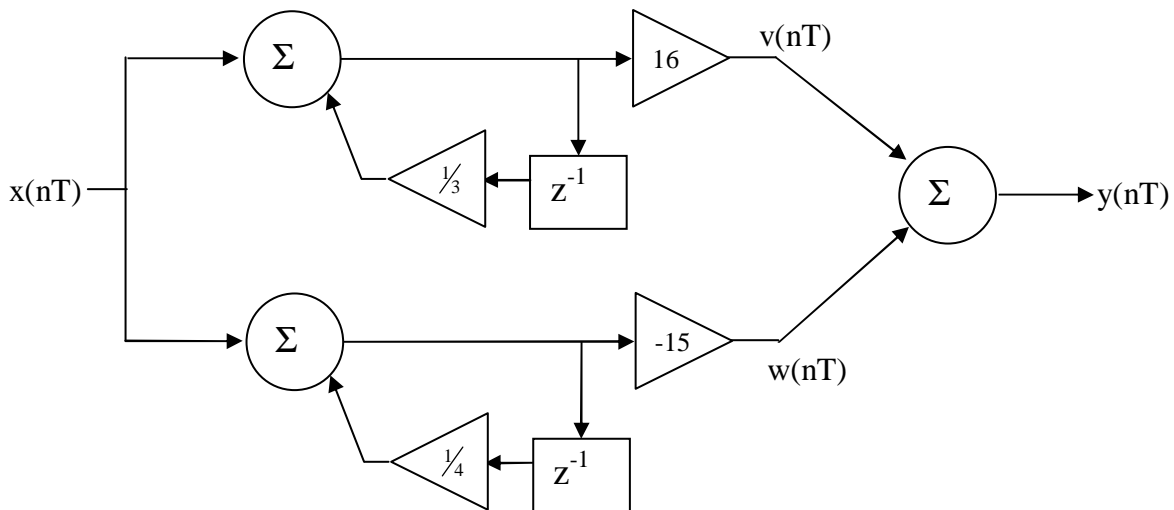


4. $H(z) = \frac{16}{(1-\frac{1}{3}z^{-1})} - \frac{15}{(1-\frac{1}{4}z^{-1})}$ Partial Fraction \Rightarrow Parallel Form

$y(nT) = v(nT) + w(nT)$

$v(nT) = 16x(nT) + (1/3)v(nT-T)$

$w(nT) = -15x(nT) + (1/4)w(nT-T)$



Just like the FIR, cascade (and parallel) forms are more resilient against quantization noise. Also, it is easier to build due to the regular patterns. Parallel form also has the advantage of having low delay. The drawback of cascade and parallel forms is that it needs factorization of the denominator.