8-4B Steady-State Frequency Response of a Linear Discrete-Time System

In this section, we study various properties of the discrete-time Fourier Transform.

As we stated earlier, it is the same as the continuous-time Fourier Transform of the sampled signal $x_s(t) = \sum_{n=0}^{\infty} x(nT)\delta(t-nT)$:

![Diagram showing the DTFT](image)

We have also claimed that it can be computing by evaluating the Z-transform around the unit circle (provided that the unit circle is inside the ROC)

**Definition of DTFT:**

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}} = \sum_{n=0}^{\infty} x(nT)e^{-j\omega nT}$$

Note that we use $X(e^{j\omega})$ to indicate the substitution $z=e^{j\omega}$.

![Diagram showing the Z-transform](image)

Let’s first show that these two definitions are equivalent. Start with the continuous-time Fourier Transform of the sampled signal:

$$X_s(j\omega) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} x(nT) \delta(t-nT)e^{-j\omega t} dt = \sum_{n=0}^{\infty} x(nT) \int_{-\infty}^{\infty} e^{-j\omega t} \delta(t-nT) dt = \sum_{n=0}^{\infty} x(nT)e^{-j\omega nT} = X(e^{j\omega})$$

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The primary reason to study Fourier Transform in continuous-time linear system is that if the input to a continuous-time linear system is a complex sinusoid of frequency $\omega_0$, the output is also a complex sinusoid of frequency $\omega_0$ with a phase shift and a gain governed by $H(j\omega_0)$.

$$x(t) = e^{j\omega_0 t} \quad \text{Continuous-Time linear system, } H(\cdot) \quad y(t) = H(j\omega_0) e^{j\omega_0 t}$$

This form of analysis, as you know, is called the steady state analysis (steady state as complex sinusoid is not transient).

It would be nice if the DTFT can do the same for Discrete-time linear system or

$$x(nT) = e^{j\omega_0 nT} \quad \text{Discrete-Time linear system, } H(\cdot) \quad y(nT) = H(e^{j\omega_0 T} e^{j\omega_0 nT})$$

And indeed it is true:

$$y(nT) = \sum_{m=0}^{\infty} x(nT - mT) h(mT)$$

$$= \sum_{m=0}^{\infty} e^{j\omega_0 (n-m)T} h(mT)$$

$$= e^{j\omega_0 nT} \sum_{m=0}^{\infty} e^{-j\omega_0 mT} h(mT)$$

$$= e^{j\omega_0 nT} H(e^{j\omega_0 T})$$

Before we go on, let’s introduce a common representation of the DTFT based on normalized frequency.

$$X(e^{j\omega T}) = X(e^{j2\pi r})$$

$r = \omega T / 2\pi$ maps $\omega \in [0, \pi] / T$ to

$r \in [0, 0.5]$  

$r =$ normalized frequency
Remarks

1. Why called normalized frequency?
   Given the sampling frequency \( f_s = 1/T \) and the frequency \( f = \omega/2\pi \), \( r \) can be more succinctly represented as
   \[
   r = \frac{\omega T}{2\pi} = \frac{f}{f_s}
   \]
   When using normalized frequency \( r \), the DTFT is written as \( H(e^{j2\pi r}) \).

2. Why ignore the negative frequency?
   For real-valued \( h(nT) \), we can deduce the negative frequency from the positive frequency:
   \[
   H(e^{j(-\omega)T}) = \sum_{n=0}^{\infty} h(nT)e^{j\omega T}
   \]
   \[
   = \left[ \sum_{n=0}^{\infty} h(nT)e^{-j\omega T} \right] (* = \text{conjugate})
   \]
   \[
   = H^*(e^{j\omega T}) \Rightarrow |H(e^{j\omega T})| = |H(e^{j\omega T})| & \angle H(e^{j\omega T}) = -\angle H(e^{j\omega T})
   \]
   Thus, it is sufficient to show only the positive part only.

The followings are a list of DTFT properties and common transform pairs. They can be easily deduced from the Z-transform tables.

<table>
<thead>
<tr>
<th>Table 2.2</th>
<th>Fourier Transform Theorems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sequence</td>
<td>Fourier Transform</td>
</tr>
<tr>
<td>( X[n] )</td>
<td>( X(e^{j\omega}) )</td>
</tr>
<tr>
<td>( y[n] )</td>
<td>( Y(e^{j\omega}) )</td>
</tr>
</tbody>
</table>

1. \( ax[n] + by[n] \)  \( aX(e^{j\omega}) + bY(e^{j\omega}) \)
2. \( x[n - n_d] \) \( n_d \text{ an integer} \)  \( e^{-j\omega n_d}X(e^{j\omega}) \)
3. \( e^{j\omega_0 n}x[n] \)  \( X(e^{j(\omega - \omega_0)}) \)
4. \( x[-n] \)  \( X(e^{-j\omega}) \)  \( X^*(e^{j\omega}) \text{ if } x[n] \text{ real.} \)
5. \( nx[n] \)  \( \frac{j dX(e^{j\omega})}{d\omega} \)
6. \( x[n] * y[n] \)  \( X(e^{j\omega})Y(e^{j\omega}) \)
7. \( x[n]y[n] \)  \( \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta \)

Parseval's theorem:
8. \( \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \)
9. \( \sum_{n=-\infty}^{\infty} x[n]^*y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega \)
Computing DTFT

Just like the continuous-time counterpart, it is most common to show the DTFT in terms of its amplitude response $|H(e^{j\omega})|$ and phase response $\angle H(e^{j\omega})$.

However, unlike continuous-time where we can draw asymptotic approximation (Bode plot) by letting $\omega \to \infty$, we can’t do that for DTFT as it is periodic.

Even though there are geometric techniques to draw the frequency responses based on the locations of poles and zeros, they are beyond the scope of this course. Here we settle with plotting the response using Matlab.
Example 1: A simple delay $H(z) = z^{-1}$

To get the magnitude and phase response, we can use the matlab routine `freqz`. In `freqz`, the Z-transform is specified by the denominator and numerator coefficients:

$$H(z) = \frac{b(1) + b(2)z^{-1} + \ldots + b(m+1)z^{-m}}{a(1) + a(2)z^{-1} + \ldots + a(n+1)z^{-n}}$$

```matlab
>> [h,w] = freqz([0 1],[1]); % Assume T=1
>> plot(w,abs(h)) % Gives Amplitude Response
```

![Amplitude Response Graph]

```matlab
>> % Gives Phase Response; Unwrap removes phase jumps
>> plot(w,unwrap(angle(h)))
```

![Phase Response Graph]

Notice that the amplitude is flat (gain =1) and the phase is linear. A system with response like this is called *distortion-less* as they essentially keep the input intact.

The negative slope of the phase, called the *group delay*, measures the delay of the frequency component. In this case,

$$-\frac{d}{d\omega} \angle H(e^{j\omega}) = -\frac{d}{d\omega}(-\omega T) = T$$

i.e. One sample for all frequencies.
Example 2: General \textit{Linear Phase System} \( H(z) = 0.1 + 0.2z^{-1} + 0.4z^{-2} + 0.2z^{-3} + 0.1z^{-4} \)

\begin{verbatim}
>> [h, w] = freqz([0.1 0.2 0.4 0.2 0.1], [1]);
>> plot(w, abs(h))
>> plot(w, unwrap(angle(h)))
\end{verbatim}

The amplitude and phase responses are

![Amplitude Response](image1)
![Phase Response](image2)

Notice that the phase is linear with group delay = 2T (2 samples) for all frequencies. It can be shown that the impulse response must be \textit{symmetric} about the middle sample if it has linear phase. Indeed, it is true for our filter:

![Impulse Response](image3)

Linear-phase filter is very important in audio and image applications because

1. In audio, the perception of chords requires different frequencies to register at the same time instance. A non-linear phase filter delays those frequencies by different amount making the chord perception dispersed.

2. In image, color edges require spatial cycles to locate precisely at a particular location. A non-linear phase filter distorts the edges by fattening them.

Can a filter have zero delay (phase)?

Yes, but such a filter MUST BE ACAUSAL! For example:
Due to the symmetry requirement, a casual IIR filter can never be linear phase. See the following example.

Example 3: a simple zero $Y(z) = \frac{1+0.5z^{-1}}{2.1-0.63z^{-1}+0.042z^{-2}}$

```
>> [h,w] = freqz([1 0.5], [2.1 -0.63 0.042]);
>> plot(w, abs(h))
>> plot(w, unwrap(angle(h)))
```

![Amplitude Response](image)

![Phase Response](image)

Even though the amplitude response is similar to that of example 2, this IIR filter gives a highly non-linear phase distortion.

Example 4: A well-designed IIR low-pass filter can provide close to linear-phase performance at least for the pass-band. More in Chapter 9.

```
>> [b,a] = butter(2,0.3);
>> [h,w] = freqz(b,a);
>> plot(w, abs(h))
>> plot(w, unwrap(angle(h)))
```

![Amplitude Response](image)

![Phase Response](image)
Inverse DTFT

There are three approaches to recover \( x(nT) \) from \( X(e^{j\omega T}) \).

**Approach 1: Table lookup or convert back to Z-transform using** \( z = e^{j\omega T} \) or \( z = e^{j\omega} \)

**Example:** \( H(e^{j2\pi}) = \frac{1}{1 - 0.5e^{-j2\pi}} \)

Using the substitution: \( z = e^{j2\pi} \), we have \( H(z) = \frac{1}{1 - 0.5z^{-1}} \)

Applying inverse Z-transform, we get \( h(nT) = (0.5)^n u(nT) \)

**Approach 2: Explicit Inverse DTFT**

Recall the definition of DTFT:

\[
X(e^{j\omega T}) = \sum_{n=0}^{\infty} x(nT)e^{-j\omega nT}
\]

Compare this with the Fourier series representation of a periodic signal \( y(t) \):

\[
y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_0 t}
\]

They bear a strong resemblance:

- \( y(t) \) is periodic in \( t \) with period \( = 2\pi/\omega_0 \) \( \leftrightarrow \)
- \( X(e^{j\omega T}) \) is periodic in \( \omega \) with period \( = 2\pi/T \)

Thus, we recognize DTFT is in fact the Fourier series representation in \( \omega \) (not \( t \)) of \( X(e^{j\omega T}) \) with \( x(nT) \) as the Fourier series coefficients!

In time-domain, we can compute the Fourier coefficients

\[
Y_k = \frac{\omega_0}{2\pi} \int_{-\omega_0}^{\omega_0} y(t)e^{-jk\omega_0 t} dt
\]

Similarly, we can compute \( x(nT) \) using the following formula:

\[
x(nT) = \frac{T}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega T})e^{j\omega nT} d\omega
\]

or with normalized frequency \( x(nT) = \int_{-0.5}^{0.5} X(e^{j2\pi r})e^{jn2\pi r} dr \)

**Approach 3: Numerical approximation by first sampling the spectrum with N points and then performing inverse Discrete Fourier Transform.**

We will discuss this method in Chapter 10.
Example: Find $h(nT)$ whose DTFT is $H(e^{j2\pi n}) = j2\pi$.

Using the inverse formula we have

$$h(nT) = \int_{-0.5}^{0.5} (j2\pi)e^{jn2\pi} dr$$

Integrating by parts yield

$$h(nT) = j2\pi \left[ \frac{r}{j2\pi} e^{jn2\pi} \right]_{r=0.5}^{r=-0.5} - \frac{1}{j2\pi} \int_{-0.5}^{0.5} e^{jn2\pi} dr$$

This becomes

$$h(nT) = \frac{1}{n} \left[ \frac{1}{2} e^{jn\pi} + \frac{1}{2} e^{-jn\pi} \right] - \frac{1}{j2\pi} \left( e^{jn\pi} - e^{-jn\pi} \right)$$

$$= \frac{1}{n} \left[ \cos(n\pi) - \frac{\sin(n\pi)}{\pi n} \right]$$

$$= \frac{\cos(n\pi)}{n} - \frac{\sin(n\pi)}{\pi n^2}$$

Further simplification can be done:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos(n\pi)/n$</td>
<td>1</td>
<td>-1/n</td>
<td>1/n</td>
<td>-1/n</td>
<td>1/n</td>
<td>-1/n</td>
<td>1/n</td>
<td>-1/n</td>
<td>1/n</td>
</tr>
<tr>
<td>$\sin(n\pi)/\pi n^2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We can see that $\frac{\cos(n\pi)}{n} = \begin{cases} 1 & n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}$ and $\frac{\sin(n\pi)}{\pi n^2} = \delta(n)$.

Thus,

$$h(nT) = \frac{\cos(n\pi)}{n} - \frac{\sin(n\pi)}{\pi n^2}$$

$$= \begin{cases} 0 & n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}$$