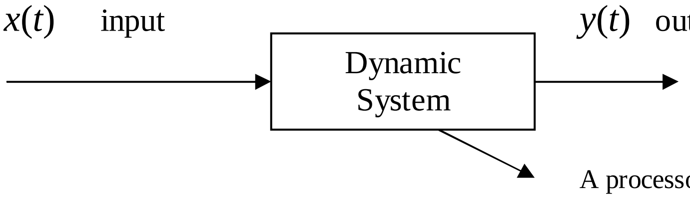
Chapter 5 The Laplace Transform

5-1 Introduction

(1) System analysis

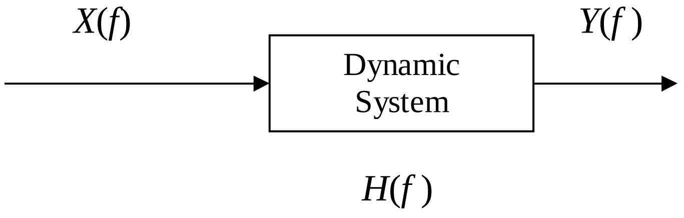


A processor which processes the input signal to produce the output

Linear Dynamic system:

$$\frac{dy^{(n)}(t)}{dt^n}+a_1\frac{dy^{(n-1)}(t)}{dt^{n-1}}+\dots+a_n y(t)=b_0\frac{dx^{(m)}(t)}{dt^m}+\dots+b_m x(t)$$

Question: Can we determine *y(t)* for a given *x(t)*?
Answer: Use Fourier transform to convert the ODE into algebraic equation!



$$Y(f)=H(f)X(f)$$
$$y(t)=F^{-1}(Y(f))$$

(2) Two Problems

1. Some common signals do not have a Fourier Transform !

   Example: What is the Fourier transform of $x(t)=e^t u(t)$?

   $X(f)=\int_{-\infty}^{\infty} e^t u(t)e^{-j2\pi ft}dt=\int_{0}^{\infty} e^t e^{-j2\pi ft}dt$ does not exist for any f as e^t blows up when $t\rightarrow\infty$.

   Even though $x(t)$ grows unbounded as $t\rightarrow\infty$, it may still exist as an intermediate step in a larger system. Consider the output when $x(t)$ is fed into a system with impulse response $h(t)=\left(3e^{-2t}-2e^{-t}\right)u(t)$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{t-\tau}u(t-\tau)\left(3e^{-2\tau}-2e^{-\tau}\right)u(\tau)d\tau \\ &= \int_{0}^{t} e^{t-\tau}\left(3e^{-2\tau}-2e^{-\tau}\right)d\tau \\ &= \left(e^{-t}-e^{-2t}\right)u(t) \end{aligned}$$

which certainly decays to 0 as $t \to \infty$.

2. Initial Condition Problem
   Say we know the output y(t) of the above dynamic system is 5 at t=0. Nowhere in the Fourier system equations below we could insert this information:

$$Y(f) = H(f)X(f)$$

$$y(t) = F^{-1}(Y(f))$$

(3) Solution: Laplace Transform

Even though $x(t)$ does not go to zero (when $t \to \infty$), but $x(t)e^{-\sigma t}$ may for large enough α.

- ⇨ We will assume all signals are "causal" : $x(t) = 0$ for $t < 0$
- ⇨ Fourier transform of $x(t)e^{-\sigma t}$:

$$\begin{aligned} \int_0^\infty (x(t)e^{-\sigma t})e^{-j\omega t}dt &= \int_0^\infty x(t)e^{-(\sigma+j\omega)t}dt \\ &= \int_0^\infty x(t)e^{-st}dt \quad \text{where } s = \sigma + j\omega \\ &= \text{Laplace Transform of } x(t) \end{aligned}$$

- We will see how Laplace transform takes care of initial conditions later.
- Even though inverse Laplace Transform exists, it involves more sophisticated concepts from complex number theory. Just like Inverse Fourier Transform, we will just use table (and Matlab).
- Laplace transform is just as nice as Fourier Transform:

$$Y(s) = H(s)X(s) + \textit{initial conditions}$$

$$y(t) = L^{-1}(Y(s))$$

- Fourier Transform of $x(t)u(t)$ can obtained by substituting $s = j\omega$ (i.e. setting $\sigma = 0$) in $X(s)$.[1].

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[1] If the Region of Converge of X(s) does not include the imaginary axis ($s = j\omega$), then its Fourier Transform does not exist. (More later)

5-2 Examples of Evaluating Laplace Transforms using the definition

(1) Step function $x(t)=u(t)$

$$\begin{aligned}
L[u(t)] &= \int_0^\infty e^{-st}dt \\
&= -\frac{1}{s}\int_0^\infty e^{-st}d(-st) \\
&= \left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=\infty} \\
&= -\frac{1}{s}\lim_{t\to\infty}\left(e^{-st}\right)+\frac{1}{s} \\
&= -\frac{1}{s}\lim_{t\to\infty}\left(e^{-\mathrm{Re}(s)t}e^{-j\,\mathrm{Im}(s)t}\right)+\frac{1}{s} \\
&= \begin{cases} 1/s & \text{if } \mathrm{Re}(s)>0 \text{ as } e^{-\mathrm{Re}(s)t}\xrightarrow{t\to\infty}0 \\ \infty & \text{if } \mathrm{Re}(s)<0 \text{ as } e^{-\mathrm{Re}(s)t}\xrightarrow{t\to\infty}\infty \\ \text{not sure} & \text{if } \mathrm{Re}(s)=0 \text{ as } \lim_{t\to\infty} e^{-j\,\mathrm{Im}(s)t} = ? \end{cases}
\end{aligned}$$

When Re(s) = 0, define ω=Im(s). The integral becomes $\int_0^\infty e^{-j\omega t}dt = \int_{-\infty}^\infty u(t)e^{-j\omega t}dt$ which is the Fourier transform of u(t). From chapter 4, we know that

$$F[u(t)] = \frac{1}{j\omega} + 2\pi\delta(\omega)$$

Note that for ω≠0, the Fourier Transform can be evaluated by substituting s=jω in the expression 1/s.

(2) Exponential $x(t)=e^{-\alpha t}u(t)$

$$\begin{aligned}
L[e^{-\alpha t}u(t)] &= \int_0^\infty e^{-\alpha t}e^{-st}dt \\
&= \int_0^\infty e^{-(s+\alpha)t}dt \\
&= \frac{-1}{s+\alpha}\int_0^\infty e^{-(s+\alpha)t}d-(s+\alpha)t \\
&= \frac{-1}{s+\alpha}\lim_{t\to\infty} e^{-(s+\alpha)t}+\frac{1}{s+\alpha} \\
&= -\frac{1}{s+\alpha}\lim_{t\to\infty}\left(e^{-\mathrm{Re}(s+\alpha)t}e^{-j\,\mathrm{Im}(s+\alpha)t}\right)+\frac{1}{s+\alpha} \\
&= \begin{cases} 1/(s+\alpha) & \text{if } \mathrm{Re}(s)>-\mathrm{Re}(\alpha) \text{ as } e^{-\mathrm{Re}(s+\alpha)t}\xrightarrow{t\to\infty}0 \\ \infty & \text{if } \mathrm{Re}(s)<-\mathrm{Re}(\alpha) \text{ as } e^{-\mathrm{Re}(s+\alpha)t}\xrightarrow{t\to\infty}\infty \\ 1/(s+\alpha)+2\pi\delta(\mathrm{Im}(s)) & \text{if } \mathrm{Re}(s)=-\mathrm{Re}(\alpha) \text{ as } \lim_{t\to\infty} e^{-j\,\mathrm{Im}(s+\alpha)t} = ? \end{cases}
\end{aligned}$$

(3) $x(t)=\delta(t)$ Recall that the impulse function is represented as a limit of convention functions straddling the origin. To incorporate the "full" delta function, we define the lower limit of our Laplace integral to be 0^-.

$$L[\delta(t)]=\int_{0}^{\infty}\delta(t)e^{-st}dt$$

$$=e^{-st}\Big|_{t=0^-}=e^{-\sigma t}e^{-j\omega t}\Big|_{t=0^-}$$

$$=e^{-\sigma t}(\cos\omega t-j\sin\omega t)\Big|_{t=0^-}$$

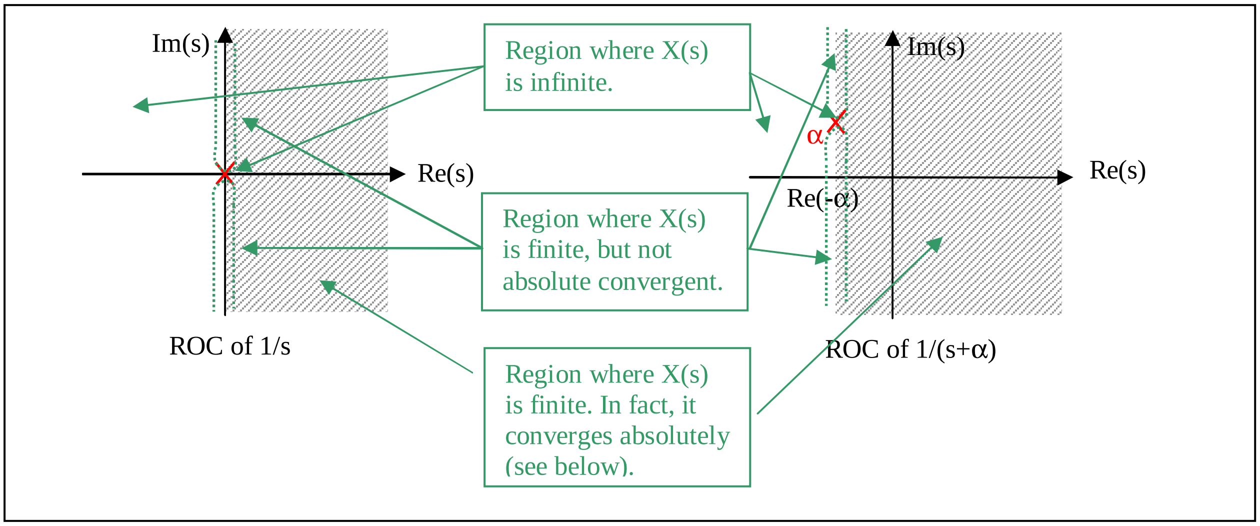
$$=1$$

No constraint on s.

5-2B Region of Convergence (ROC)

ROC: Pictorial description of the value s where the Laplace Transform of a function exists.

Anatomy of ROC



Properties of ROC

1. It is bordered by the RIGHTMOST POLES of $X(s)$. A pole is defined by the complex value s' such that the algebraic $X(s')$ is infinite.
2. If s' is in the interior of the ROC, then all s with $\mathrm{Re}(s)=\mathrm{Re}(s')$ are inside the ROC.
3. It extends to positive infinity

In the interior of ROC (not boundary), not only does the Laplace integral converge, it converges ABSOLUTELY.

Normal convergent (or X(s) is finite): $X(s) = \int_0^\infty x(t)e^{-st}dt < \infty$

Absolute convergent: $\int_0^\infty |x(t)e^{-st}|dt < \infty$

We cannot prove that without venturing into a branch of mathematics called the complex variable theory.

However, knowing that X(s) converges absolutely inside the ROC, we can explain some of the properties:

For example:

2. If *s'* is in the interior of the ROC, then all *s* with Re(*s*)=Re(*s'*) are inside the ROC.

$$\int_0^\infty \left|x(t)e^{-st}\right|dt = \int_0^\infty \left|x(t)\exp(-\operatorname{Re}(st) - j\operatorname{Im}(st))\right|dt$$

Why?

$$= \int_0^\infty \left|x(t)\exp(-\operatorname{Re}(st))\right| \cdot \left|\exp(-j\operatorname{Im}(st))\right|dt$$

Note that $\left|\exp(-j\operatorname{Im}(st))\right| = \sqrt{\cos^2(\operatorname{Im}(st)) + \sin^2(\operatorname{Im}(st))} = 1$, we have

$\int_0^\infty \left|x(t)e^{-st}\right|dt = \int_0^\infty \left|x(t)\exp(-\operatorname{Re}(st))\right|dt$ which does not depend on Im(s).

3. ROC extends to positive infinity.

Assume p is in the ROC. Let q be a complex number with $\operatorname{Re}(q) > \operatorname{Re}(p)$.

$$\int_0^\infty \left|x(t)e^{-qt}\right|dt = \int_0^\infty |x(t)|\left|\exp(-\operatorname{Re}(qt))\right|dt \le \int_0^\infty |x(t)|\left|\exp(-\operatorname{Re}(pt))\right|dt < \infty.$$

The second last inequality is due to the fact that $|\exp(-\operatorname{Re}(qt))| \le |\exp(-\operatorname{Re}(pt))|$. Thus *q* must also be in the ROC.

Example:

$X(s) = \dfrac{(s-4)^2}{(s+1)(s+3)}$ has two poles at -1 and -3 as well as a double zero at 4. Its ROC is {s: s>-1}.

Final note:

Fourier Transform: if the imaginary axis ($s = j\omega$) is entirely inside the ROC of $X(s)$, then the Fourier Transform $X(f)$ exists. If $s = j\omega$ is entirely outside, $X(f)$ does not exist. If $s = j\omega$ is the boundary of the ROC, the Fourier transform must be evaluated by other means due to the presence of poles (e.g. Try x(t) = u(t))

Note: Using matlab to find Laplace Transform

```
>> x1 = sym(’cos(omega*t)’)
x1  =
cos(omega*t)

>> X1 = laplace(x1)                % Laplace Transform
X1  =
s/(s^2+omega^2)

>> pretty(X1)
                   s
               ---------
                2      2
               s +omega

>> X2 = sym(’omega/(s^2+omega^2)’)
X2  =
omega/(s^2+omega^2)

>> x2 = ilaplace(X2)               % Inverse Laplace
x2  =
sin(omega*t)

>> dirac(0)                        % Delta Function
ans =
inf
>> dirac(0.45)
ans =
0
>> laplace(sym(’dirac(t)’))
ans =
1

>> heaviside(0)                    % Step Function
ans =
NaN
>> heaviside(-13)
ans =
0
>> heaviside(1)
ans =
1
>> laplace(sym(’heaviside(t)’))
ans =
1/s
```