5-3 Inversion of Rational Functions

All the Laplace Transform you will encounter has the following form:

$$\underbrace{\frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots a_{n-1}s + a_n}}_{\text{Rational function } X(s)} \underbrace{e^{-\tau s}}_{\text{Delay}}$$

Why? Rational functions come out naturally from the Laplace transform of ordinary differential equations, exponential, cosine and sine functions.

Our strategy: *breakdown a general rational function into simpler fractions and polynomials whose inverse transforms have already been computed in Table 5-3.*

How?

I. Convert non-proper rational function into proper rational function
Non-proper: degree of numerator ≥ degree of denominator

Approach: Long Division
Example:

$$X(s) = \frac{s^5 + 5s^4 + 9s^3 + 9s^2 + 12s + 4}{s^4 + 4s^3 + 5s^2 + 4s + 4}$$

$$\begin{array}{r l}
 & s + 1 \\
s^4 + 4s^3 + 5s^2 + 4s + 4 \,\big) & s^5 + 5s^4 + 9s^3 + 9s^2 + 12s + 4 \\
 & \underline{s^5 + 4s^4 + 5s^3 + 4s^2 + 4s} \\
 & \quad s^4 + 4s^3 + 5s^2 + 8s + 4 \\
 & \quad \underline{s^4 + 4s^3 + 5s^2 + 4s + 4} \\
 & \qquad 4s
\end{array}$$

$$X(s) = \underbrace{s + 1}_{\text{Polynomial}} + \underbrace{\frac{4s}{s^4 + 4s^3 + 5s^2 + 4s + 4}}_{\text{Proper fraction}}$$

It is easy to find the Laplace transform of polynomials:

$$s^n \leftrightarrow \delta^{(n)}(t)$$

$$L^{-1}(s+1) = L^{-1}(s) + L^{-1}(1) = \delta^{(1)}(t) + \delta(t)$$

II. Factorize the denominator polynomial:

Example:

$$s^4 + 4s^3 + 5s^2 + 4s + 4 = (s+2)^2(s^2+1) = (s+2)^2(s+j)(s-j)$$

- Factorize into REAL linear and irreducible quadratic factors. Further break down REAL irreducible quadratic factors into conjugate roots.
- Notice some factors may repeat.
- No analytical formula for polynomial with degree 5 or other. Need to rely on numerical methods.

```
>> roots([1 4 5 4 4])
ans =
  -2.0000
  -2.0000
   0.0000 + 1.0000i
   0.0000 - 1.0000i
>> factor(sym('s^4+4*s^3+5*s^2+4*s+4'))
ans =
(s+2)^2*(s^2+1)
```

III. Break down the proper rational function into simple fractions. This technique is called *Partial Fraction Expansion.*

Example: $X(s) = \dfrac{4s}{(s+2)^2(s^2+1)}$

First, write down the full partial fraction expansion with enough number of unknowns to represent all possible numerators:

$$X(s) = \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{d}{s+j} + \frac{\bar{d}}{s-j}$$

Four simple rules to write down unknowns:

1. $\dfrac{\ldots}{(s-a)\ldots} \to \dfrac{k}{s-a} + \ldots$ for real root a
2. $\dfrac{\ldots}{(s-a)(s-\bar{a})\ldots} \to \dfrac{k}{s-a} + \dfrac{\bar{k}}{s-\bar{a}} + \ldots$ for complex conjugate roots a and $\bar{a}$
3. $\dfrac{\ldots}{(s-a)^n\ldots} \to \dfrac{k_1}{s-a} + \dfrac{k_2}{(s-a)^2} + \ldots + \dfrac{k_n}{(s-a)^n} + \ldots$
4. $\dfrac{\ldots}{(s-a)^n(s-\bar{a})^n\ldots} \to \dfrac{k_1}{s-a} + \dfrac{\bar{k_1}}{s-\bar{a}} + \ldots + \dfrac{k_n}{(s-a)^n} + .\dfrac{\bar{k_n}}{(s-a)^n} + \ldots$

IV. Solve for the unknowns.

Many methods exist. I will talk about two:

1. Compare coefficients (The Dumb Way)

Example: $X(s) = \dfrac{4s}{(s+2)^2(s^2+1)}$

$$\begin{aligned} X(s) &= \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{d}{s+j} + \frac{\bar{d}}{s-j} \\ &= \frac{(a+2\,\mathrm{Re}(d))s^3 + (2a+b+8\,\mathrm{Re}(d)+2\,\mathrm{Im}(d))s^2 + (a+8\,\mathrm{Re}(d)+8\,\mathrm{Im}(d))s + (2a+b+8\,\mathrm{Im}(d))}{(s+2)^2(s^2+1)} \end{aligned}$$

Notice that when you expand it out, no complex coefficients remain and you can ALWAYS do that if you follow the four simple rules above.

You can ask Matlab to do this too:

```
>> x=sym('a/(s+2)+b/((s+2)^2)+(rd+i*id)/(s+i)+(rd-i*id)/(s-i)');

>> x1 = simplify(x)                      % Multiply out
x1 =
(8*id*s+2*id*s^2+a*s+a*s^3+2*a*s^2+2*a+b*s^2+b+8*rd*s^2+8*rd*s+2
*rd*s^3+8*id)/(s+2)^2/(s^2+1)

>> x2 = sort(collect(x1)) %collect like terms in descending power
x2 =
((a+2*rd)*s^3+(2*a+b+8*rd+2*id)*s^2+2*a+(a+8*rd+8*id)*s+b+8*id)/
(s+2)^2/(s^2+1)
```

Since we are given $X(s)=\frac{4s}{(s+2)^2(s^2+1)}$, we know that the numerator polynomial is just $4s$. By comparing coefficients, we get the following set of equations:

$$\begin{aligned}
a+2\,\mathrm{Re}(d) &= 0\\
2a+b+8\,\mathrm{Re}(d)+2\,\mathrm{Im}(d) &= 0\\
a+8\,\mathrm{Re}(d)+8\,\mathrm{Im}(d) &= 4\\
2a+b+8\,\mathrm{Im}(d) &= 0
\end{aligned}$$

To solve this system of equations numerically, rewrite it in matrix form:

$$\begin{pmatrix}1 & 0 & 2 & 0\\ 2 & 1 & 8 & 2\\ 1 & 0 & 8 & 8\\ 2 & 1 & 0 & 8\end{pmatrix}\begin{pmatrix}a\\ b\\ \mathrm{Re}(d)\\ \mathrm{Im}(d)\end{pmatrix}=\begin{pmatrix}0\\0\\4\\0\end{pmatrix}$$

Then use matlab:

```
>> A = [1 0 2 0;2 1 8 2;1 0 8 8; 2 1 0 8];
>> b = [0;0;4;0];
>> A\b                    % Solution to Ax=b
ans =
   -0.4800
   -1.6000
    0.2400
    0.3200
```

or $a=-0.48, b=-1.6, \mathrm{Re}(d)=0.24, \mathrm{Im}(d)=0.32$

2. Heaviside's Theorem:

Alternatively, you can use the Heaviside's Theorem:

Example/ $X(s) = \dfrac{4s}{(s+2)^2(s^2+1)} = \dfrac{a}{s+2} + \dfrac{b}{(s+2)^2} + \dfrac{d}{s+j} + \dfrac{\bar{d}}{s-j}$

Multiply both sides by $(s+j)$,

$$\frac{4s}{(s+2)^2(s-j)} = \frac{a(s+j)}{s+2} + \frac{b(s+j)}{(s+2)^2} + d + \frac{\bar{d}(s+j)}{s-j}$$

Note that every term on the right has a factor (s+j).

Substitute s=- j (the root for (s+j)):

$\dfrac{4(-j)}{(-j+2)^2(-j-j)} = d$ or $d = 0.24 + j0.32$

A similar approach can solve for b as well. Multiply both sides by $(s+2)^2$:

$$\frac{4s}{(s^2+1)} = a(s+2) + b + \frac{d(s+2)^2}{s+j} + \frac{\bar{d}(s+2)^2}{s-j}$$

Substituting s=-2: $b = -1.6$

However, we cannot use this approach find a. If we multiple $(s+2)$ to both sides, we have

$$\frac{4s}{(s^2+1)(s+2)} = a + \frac{b}{(s+2)} + \frac{d(s+2)}{s+j} + \frac{\bar{d}(s+2)}{s-j}$$

As (s+2) is a repeated factor, we still have one left after the multiplication. We can't substitute s=-2 here as it will turn both sides to infinity.

Instead, we still multiply $(s+2)^2$ but then we take the derivative with respect to s:

$$\frac{d}{ds}\frac{4s(s+2)^2}{(s^2+1)(s+2)} = \frac{4}{(s^2+1)} - \frac{8s^2}{(s^2+1)^2} = a + \frac{2d(s+2)}{s+j} - \frac{d(s+2)^2}{(s+j)^2} + \frac{2\bar{d}(s+2)}{s-j} - \frac{d(s+2)^2}{(s-j)^2}$$

Taking the derivative kills b and exposes a while all the remaining terms will still have a factor of (s+2). Substitute s=-2, we get a =-0.48.

The same approach can be used for factors with multiplicity > 2.

Example: Find the partial fraction expansion of

$$X(s) = \frac{4s+1}{(s+1)^3(s+2)}$$

Again, we first write down the partial fraction expansion

$$X(s) = \frac{4s+1}{(s+1)^3(s+2)}$$

Here is the general rule: if the denominator of $X(s)$ has a factor $(s\text{-}p)^n$ then the coefficient a_m for the partial fraction $1/(s\text{-}p)^m$ for $m<n$ is

$$a_m = \frac{1}{(n-m)!}\frac{d^{n-m}}{ds^{n-m}}\left[X(s)(s-p)^n\right]_{s=p}$$

and for $m=n$

$$a_n = \left[X(s)(s-p)^n\right]_{s=p}$$

Both methods arrive at the same answer:

$$X(s) = \frac{-0.48}{s+2} + \frac{-1.6}{(s+2)^2} + \frac{0.24+j0.32}{s+j} + \frac{0.24-j0.32}{s-j}$$

The previous four steps: converting to proper polynomial, factorizing denominator polynomial, writing down partial fraction expansion, and solving for the unknown coefficients can all done by a simple MATLAB command: `residue`

From the Matlab Help-page:

```
RESIDUE Partial-fraction expansion (residues).
[R,P,K] = RESIDUE(B,A) finds the residues, poles and direct term
of a partial fraction expansion of the ratio of two polynomials
B(s)/A(s).

If there are no multiple roots,
       B(s)       R(1)       R(2)             R(n)
       ----  =  -------- + -------- + ... + -------- + K(s)
       A(s)     s - P(1)   s - P(2)         s - P(n)
….
If P(j) = ... = P(j+m-1) is a pole of multiplicity m, then the
expansion includes terms of the form
```

```
   R(j)          R(j+1)                  R(j+m-1)
-------- + ------------   + ... + ------------
s - P(j)   (s - P(j))^2            (s - P(j))^m
```

Using the sample example: $X(s)=\dfrac{s^5+5s^4+9s^3+9s^2+12s+4}{s^4+4s^3+5s^2+4s+4}$

By hand, this is the answer we get: $X(s)=s+1+\dfrac{-0.48}{s+2}+\dfrac{-1.6}{(s+2)^2}+\dfrac{0.24+j0.32}{s+j}+\dfrac{0.24-j0.32}{s-j}$

First, we define the rational functions by writing the coefficients of the denominator and numerator in DESCENDING ORDER without any SKIP OF POWER:

```
>> den = [1 4 5 4 4];
>> num = [1 5 9 9 12 4];
```

Running the residue() command:

```
>> [r,p,k] =residue(num,den)
r =
  -0.4800                   % coefficient of 1st factor
  -1.6000                   % coefficient of 2nd factor
   0.2400 - 0.3200i         % coefficient of 3rd factor
   0.2400 + 0.3200i         % coefficient of 4th factor
p =
  -2.0000                   % first factor: (s+2)
  -2.0000                   % second factor: (s+2)^2
   0.0000 + 1.0000i         % third factor: (s-j)
   0.0000 - 1.0000i         % fourth factor: (s+j)
k =
     1     1                % leading polynomial: s+1
```

V. Compute the inverse Laplace transform for each term

1. Combine conjugate roots, separate fraction for each numerator term:

$$\begin{aligned} X(s) &= s+1+\frac{-0.48}{s+2}+\frac{-1.6}{(s+2)^2}+\frac{0.24+j0.32}{s+j}+\frac{0.24-j0.32}{s-j} \\ &= s+1+\frac{-0.48}{s+2}+\frac{-1.6}{(s+2)^2}+\frac{0.48s}{s^2+1}+\frac{0.64}{s^2+1} \end{aligned}$$

2. Write every term in the forms whose Laplace transforms we know:

$$s+1 \rightarrow \delta^{(1)}(t)+\delta(t)$$

$$\frac{-0.48}{s+2} \rightarrow (-0.48)e^{-2t}$$

$$\frac{-1.6}{(s+2)^2} \rightarrow (-1.6)te^{-2t}$$

$$\frac{0.48s}{s^2+1} \rightarrow 0.48\cos t$$

$$\frac{0.64}{s^2+1} \rightarrow 0.64\sin t$$

For general quadratic term s^2+as+b, we need to complete the square $(s+a/2)^2+b-a^2/4$ (see the example property 2: complex frequency shift theorem)

3. Use linearity to combine them and you are done!!

$$x(t) = (\delta^{(1)}(t)+\delta(t)-0.48e^{-2t}-1.6te^{-2t}+0.48\cos t+0.64\sin t)u(t)$$