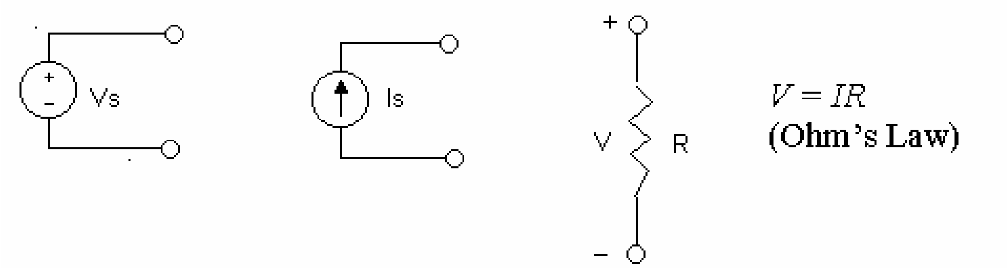
Applications of the Laplace Transform

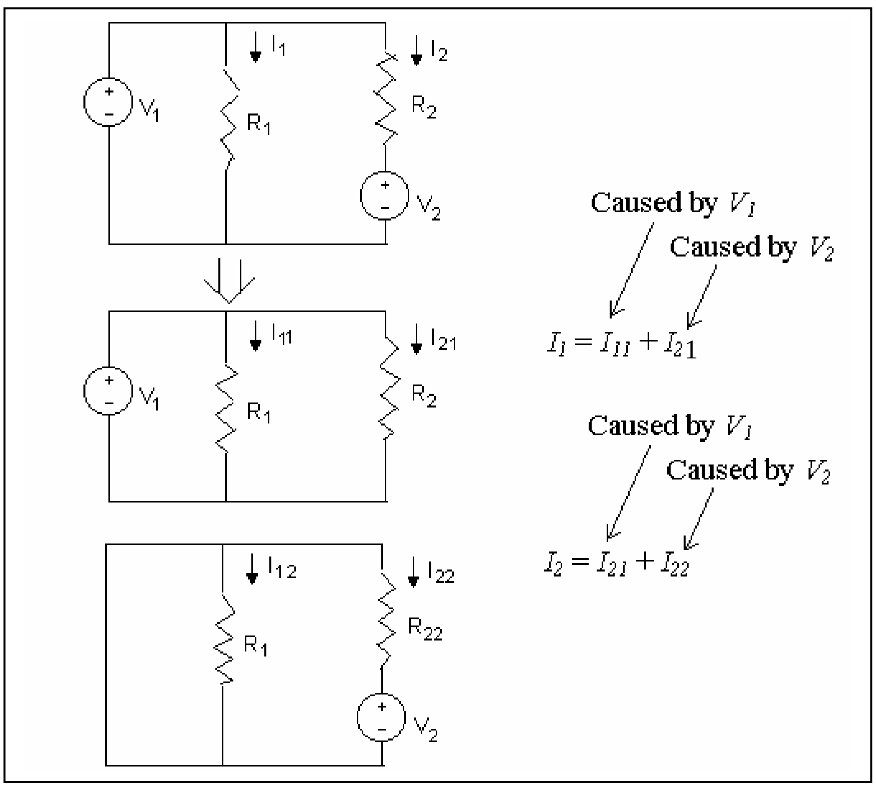
Application in Circuit Analysis

1. Review of Resistive Network

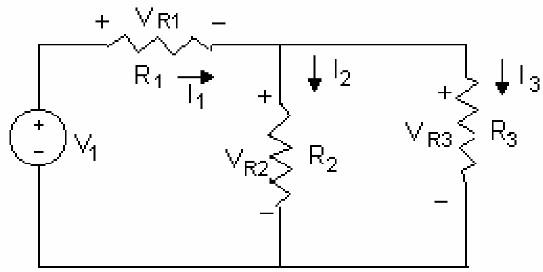
1) Elements



2) Superposition



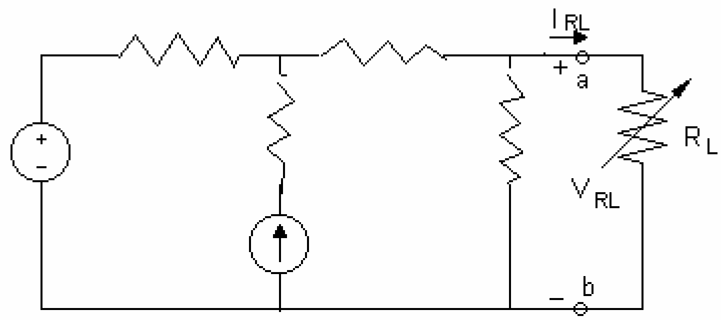
3) KVL and KCL – Select a node for ground. Watch out for signs!



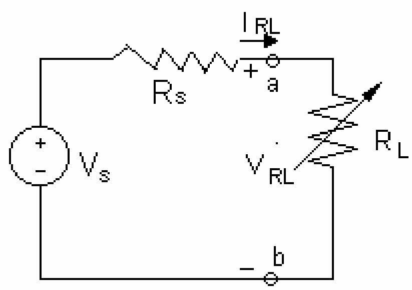
KVL:
$$V_1 = V_{R1} + V_{R2}$$
$$V_1 = V_{R1} + V_{R3}$$
$$V_{R2} = V_{R3}$$

KCL:
$$I_1 = I_2 + I_3$$

4) Equivalent Circuits

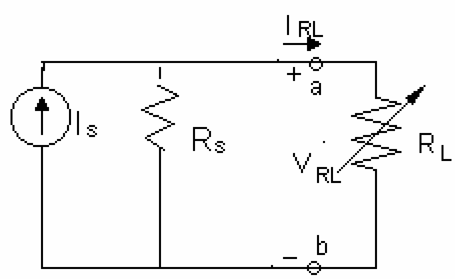


Thevenin Equivalent Circuit ⇒



$V_s = V_{OC}$ = Open Circuit Voltage
R_s = Equivalent Resistance

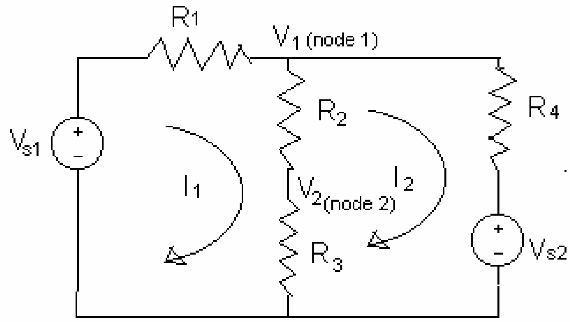
Norton Equivalent Circuit or



$I_s = I_{SC}$ = Short Circuit Current

R_s = Same as before

5) Nodal Analysis and Mesh Analysis



Nodal Analysis (Use KCL)

$$\begin{cases} \dfrac{V_{S1}-V_1}{R_1}+\dfrac{V_2-V_1}{R_2}+\dfrac{V_{S2}-V_1}{R_4}=0 \\ \dfrac{V_1-V_2}{R_2}+\dfrac{0-V_2}{R_3}=0 \end{cases}$$

Solve for V_1 and V_2 and then calculate other currents and voltages.

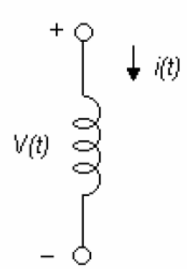
Mesh analysis (use KVL)

$$\begin{cases} V_{S1}=R_1I_1+R_2(I_1-I_2)+R_3(I_1-I_2) \\ V_{S1}=R_1I_1+R_4I_2+V_{S2} \end{cases}$$

Solve for I_1 and I_2.

2. <u>Characteristics of Dynamic Networks</u>

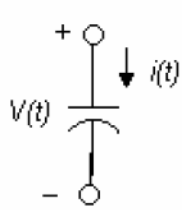
1) Inductor



$$v_L(t)=L\frac{d}{dt}i_L(t)$$

$$\text{or } i_L(t)=\frac{1}{L}\int_{-\infty}^{t}v_L(\tau)d\tau$$

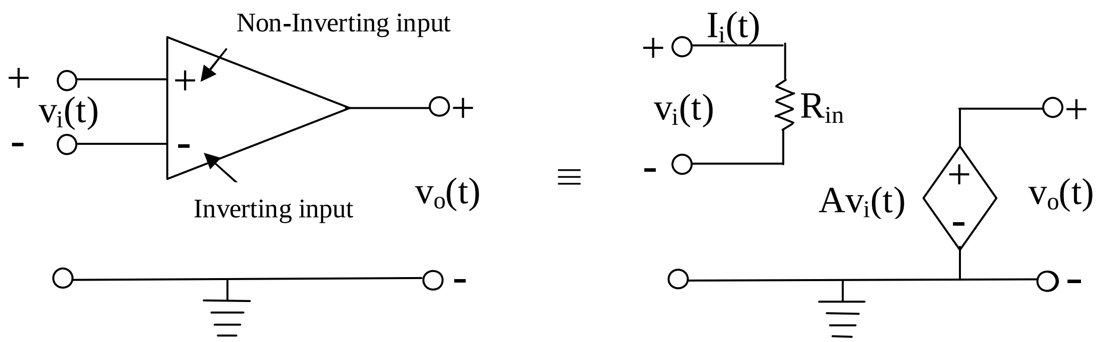
2) Capacitor



$$i_C(t)=C\frac{d}{dt}v_C(t)$$

$$\text{or } v_C(t)=\frac{1}{C}\int_{-\infty}^{t}i_C(\tau)d\tau$$

3) Operation Amplifier

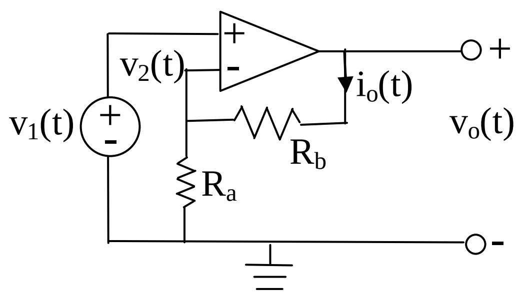


A general op-amp model is described above.

In practice, the input resistance, R_{in}, is very large ($> 10^{12}\ \Omega$) and the gain, A, is very large ($>10^5$). Thus, we will use the ideal model in the analysis:

1. Input current $I_i(t) = 0$ (due to the large input impedence)
2. Input voltage difference $v_i(t) = 0$ and output voltage $v_o(t)$ is dictated by external circuit (due to the large gain)

Example:



Based on the ideal op-amp model,

$$v_2(t) = v_1(t) \qquad (1)$$

Also, as the op-amp does not have any input current, applying KCL at the inverting port, we have

$$v_2(t)/R_a = (v_o(t)-v_2(t))/R_b$$

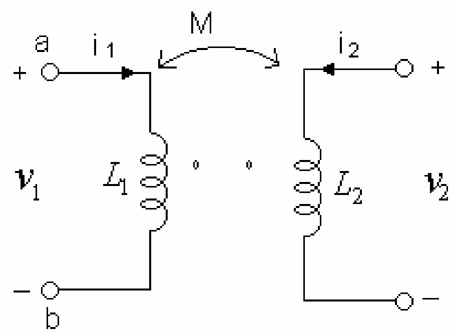
$$v_o(t)/v_2(t) = 1+R_b/R_a$$

Plug in (1), we have $$v_o(t)/v_1(t) = 1+R_b/R_a$$

This circuit is called *Non-Inverting Amplifier*.

4) Mutual Inductor – used in transformer.

Two separate circuits with coupling currents.



Basic equations

$$\begin{cases} v_1(t) = L_1 \dfrac{di_1}{dt} + M\dfrac{di_2}{dt} \\ v_2(t) = M\dfrac{di_1}{dt} + L_2\dfrac{di_2}{dt} \end{cases}$$

↑ Very easy to write, very difficult to analyze!

> Make sure both i_1 and i_2 point either away or toward the polarity marks to make the mutual inductance M positive.

To link the two circuits together, introduce a combined current term (i_1+i_2):

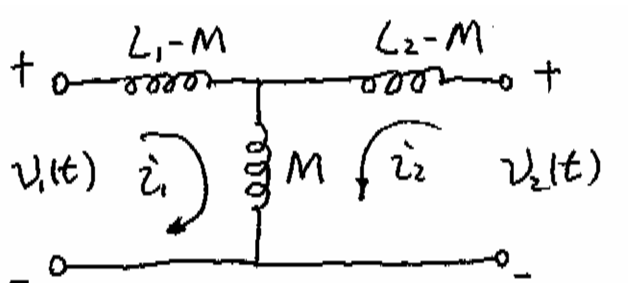
$$v_1(t) = L_1\frac{di_1}{dt} - M\frac{di_1}{dt} + M\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$= (L_1 - M)\frac{di_1}{dt} + M\frac{d}{dt}(i_1 + i_2)$$

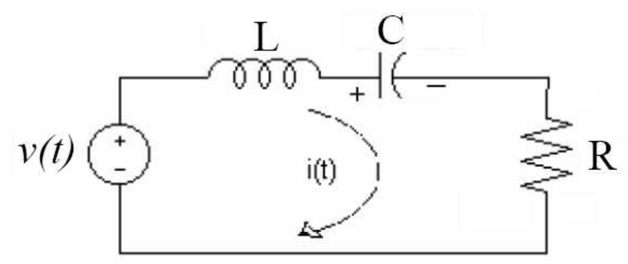
$$v_2(t) = M\frac{di_1}{dt} + M\frac{di_2}{dt} + L_2\frac{di_2}{dt} - M\frac{di_2}{dt}$$

$$= M\frac{d}{dt}(i_1 + i_2) + (L_2 - M)\frac{di_2}{dt}$$

Equivalent circuit:



Example : Apply mesh analysis to the following circuit



$$i(0) = v_c(0) = 0$$

$$v(t) = L\frac{di(t)}{dt} + \frac{1}{C}\int_{0^-}^{t} i(\tau)d\tau + Ri(t)$$

↑ **a differential equation.**

Using Laplace Transform

$$V(s) = V_L(s) + V_C(s) + V_R(s)$$

$$= L(sI(s) - i(0^-)) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{v_C(0)}{s}\right] + RI(s)$$

$$= (Ls)I(s) + \frac{1}{Cs}I(s) + RI(s)$$

Define 'Generalized Resistors' (Impedances)

$$Z_L(s) = Ls \Rightarrow V_L(s) = I(s)Z_L(s)$$

$$Z_C(s) = \frac{1}{Cs} \Rightarrow V_C(s) = I(s)Z_C(s)$$

Both capacitor and inductor behave exactly like a resistor!

$$\Rightarrow V_s(s) = Z_L(s)I(s) + Z_C(s)I(s) + RI(s)$$

$$\Rightarrow I(s) = \frac{V_s(s)}{Z_L(s) + Z_C(s) + R}$$

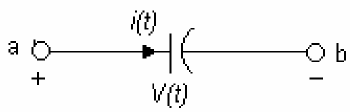
Everything we know about resistive network can be applied to dynamic network in Laplace domain:

- Generalized Ohms Law
- superposition
- KVL and KCL
- Equivalent circuit
- Nodal analysis and mesh analysis

3. Laplace transform models of circuit elements.

What if the initial conditions are not zero?

1) Capacitor



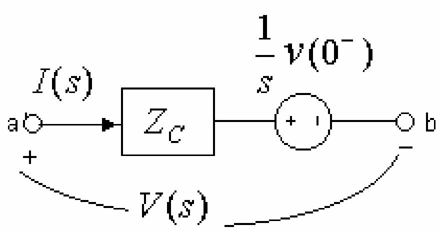
$$i(t) = C\frac{dv(t)}{dt}$$

$$I(s) = CsV(s) - Cv(0^-)$$

$$V(s) = \frac{1}{Cs}I(s) + \frac{1}{s}v(0^-)$$

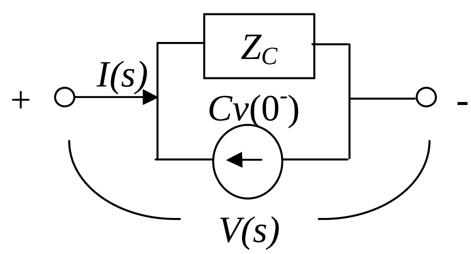
define $$Z_C(s) = \frac{1}{Cs}$$

$$V(s) = Z_C(s)I(s) + \frac{1}{s}v(0^-)$$



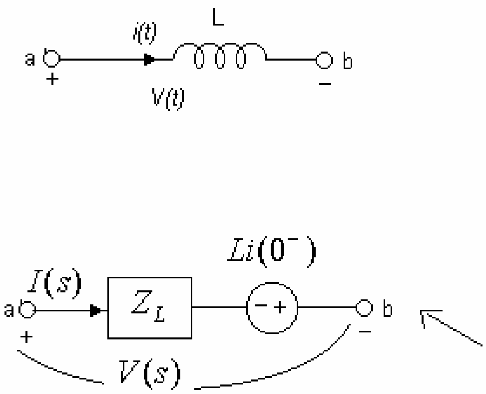
$\Rightarrow$ **model: an impedance ('generalized resistor') and a voltage source in series.**

Alternatively, you can also represent it as an impedance and a parallel current source (Norton equivalent circuit)



BE VERY CAREFUL ABOUT THE POLARITY OF VOLTAGE SOURCE AND THE DIRECTION OF CURRENT SOURCE!!

2) Inductor



$$V(t) = L\frac{di(t)}{dt}$$

$$V(s) = LsI(s) - Li(0^-)$$

'Generalized resistor' (Impedance)

$Z_L(s) = Ls$ and

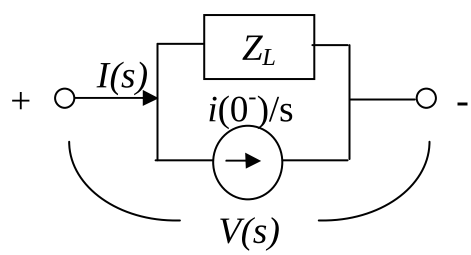
a negative voltage source $Li(0^-)$ in series.

Can we obtain $V(s) = LsI(s) - Li(0^-)$

From this circuit models?

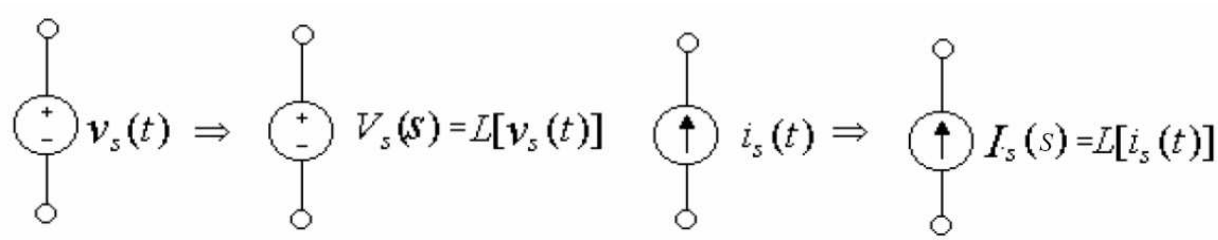
Yes!

Alternatively, you can also represent it as an impedance and a parallel current source (Norton equivalent circuit)



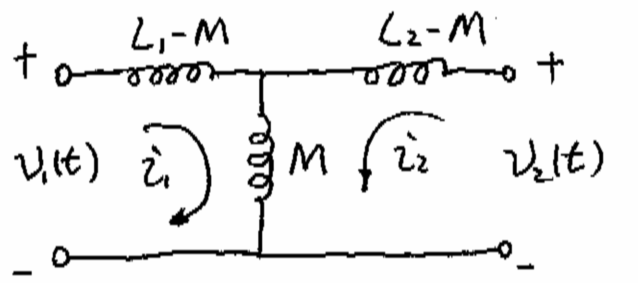
3) Resistor $V(s) = RI(s)$

4) Voltage and Current Sources (Don't forget to apply Laplace Transform on them)

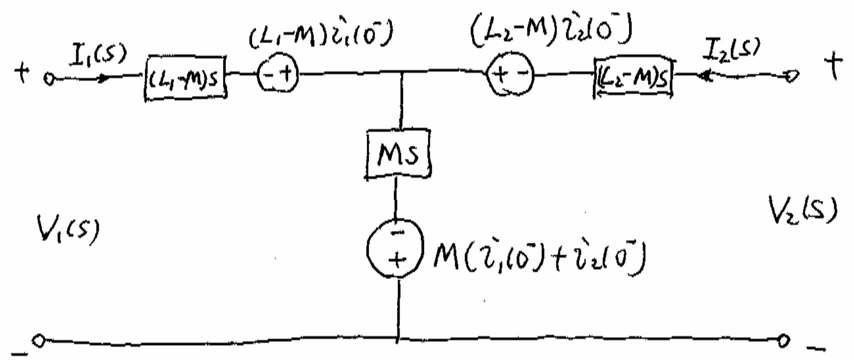


5) Op-Amp : same ideal model assumption

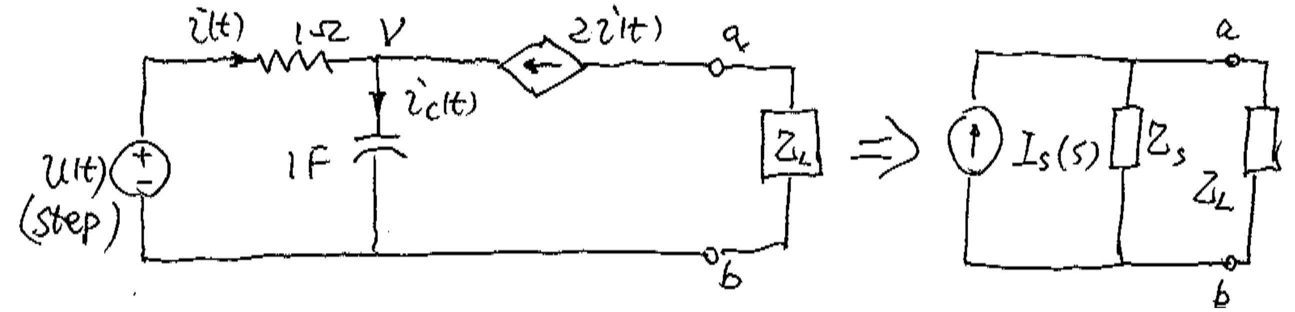
6) Mutual Inductance (Transformers)



⇓ Laplace transform model: Obtain it by using inductance model

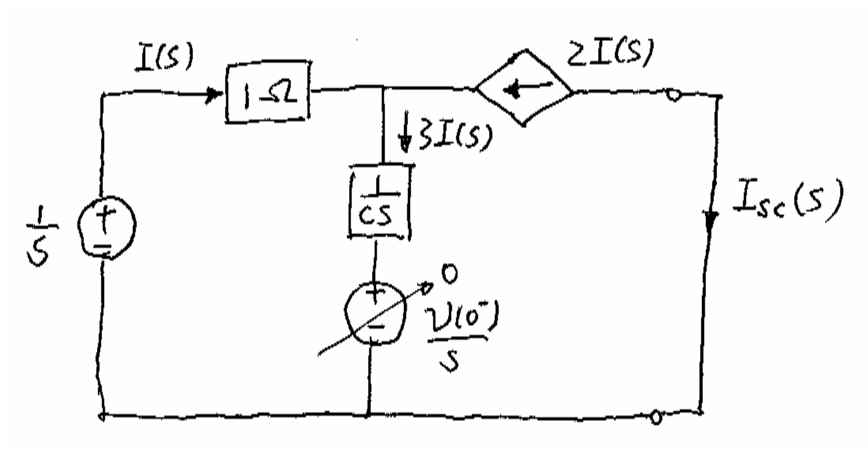


Example: Find Complex Norton Equivalent circuit given $v_c(0^-) = 0$



Solution

1) Compute the Short-Circuit Current $I_s(s) = I_{sc}$



Straightforward to see: $I_S(s) = -2I(s)$

To compute I(s), apply mesh analysis on the left loop:

$$\frac{1}{s} = I(s) \times 1\Omega + 3I(s)\frac{1}{s} = (1 + \frac{3}{s})I(s) = \frac{s+3}{s}I(s)$$

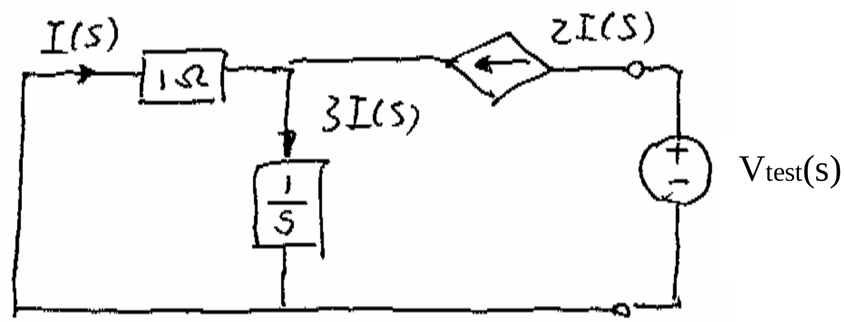
$$\Rightarrow I(s) = \frac{1}{s+3}$$

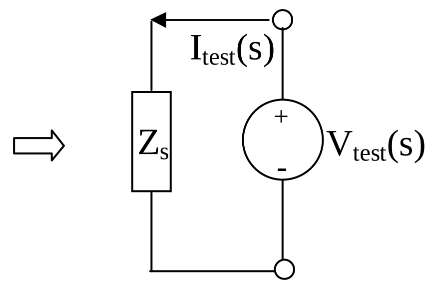
$$\Rightarrow I_s(s) = -2I(s) = -\frac{2}{s+3}$$

No need to do inverse Laplace transform as the equivalent circuit is in the s-domain.

2) Find the equivalent impedance Z_s

Normally, we can just kill all the independent sources and combine the impedances (using resistive combination rules). However, as there is a dependent source, we need to drive it with a test voltage:





$$Z_s = \frac{V_{test}(s)}{I_{test}(s)} = \frac{V_{test}(s)}{2I(s)}$$

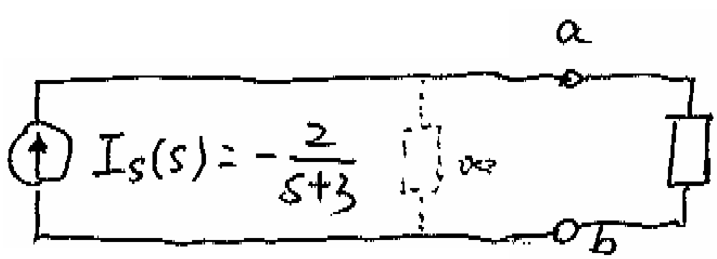
Mesh analysis on the left loop:

$$I(s)\times 1\Omega + \frac{3I(s)}{s} = 0$$

$$\Rightarrow (1\,\Omega)I(s) = -\left(\frac{3}{s}\right)I(s)$$

$$\Rightarrow I(s) = 0$$

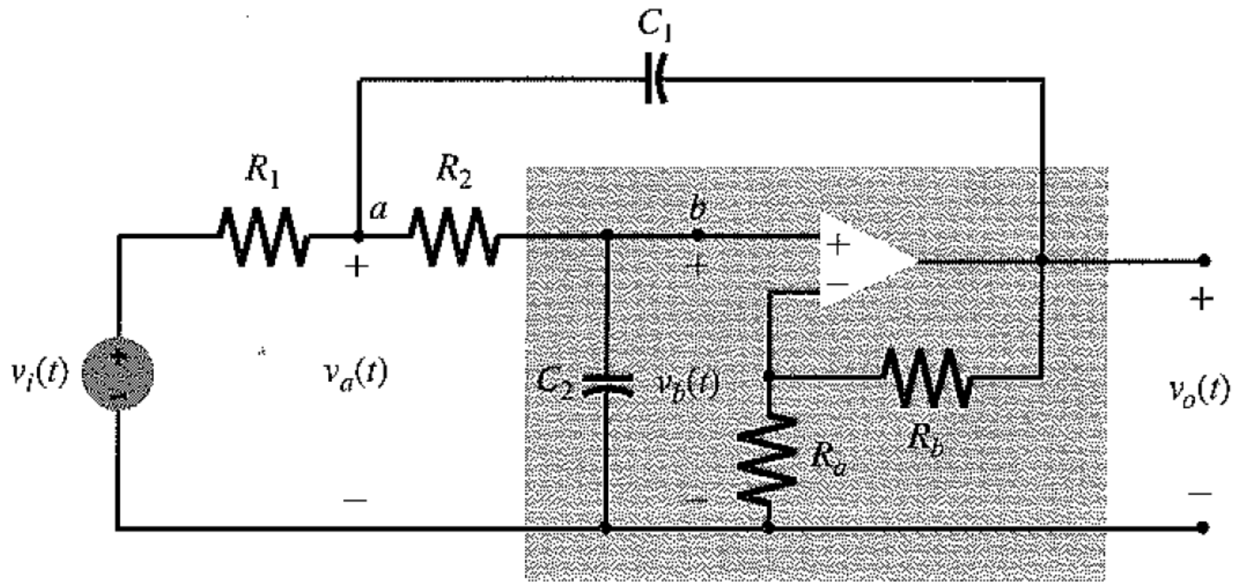
So we got an interesting result: $Z_s = \dfrac{V_{test}(s)}{0} = \infty \Rightarrow \text{OPEN CIRCUIT}$



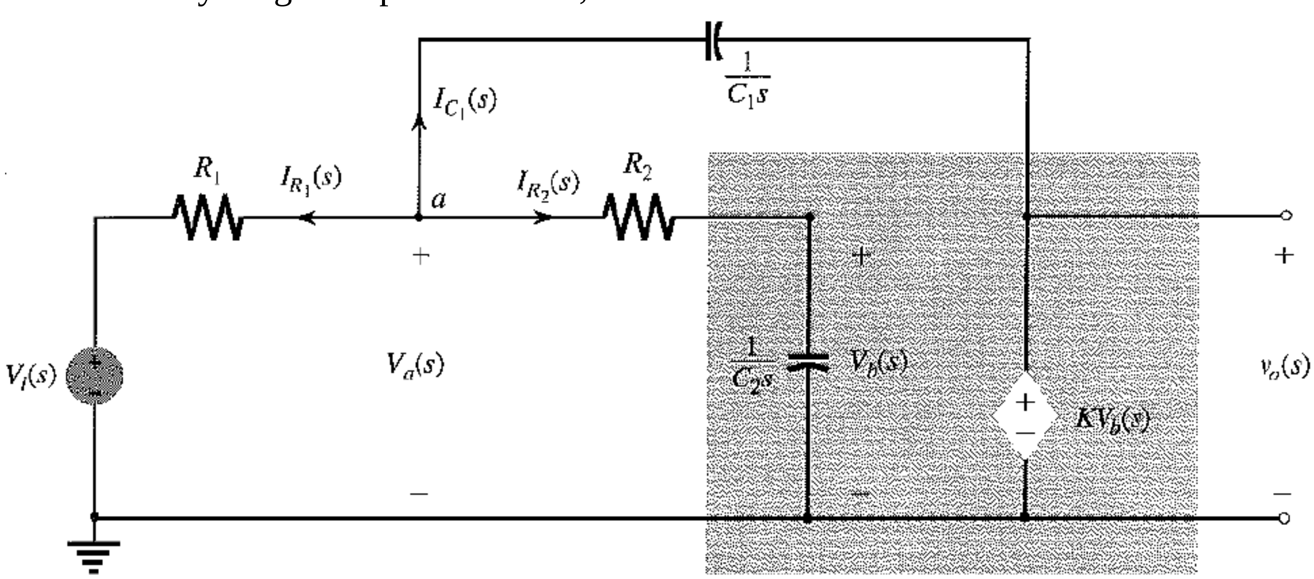
Z_L

Example: Find the transfer function H(s) = $V_o(s)/V_i(s)$ of the following circuit. Assume all initial conditions are zero.

This is called the Sallen-Key circuit, which we will see again in filter design.



Rewrite everything in Laplace domain, we have



We recognize the op-amp configuration as a non-inverting amplifier, so we have

$$K = 1 + \frac{R_b}{R_a}$$

To find V_o, we need V_b which depends on V_a.

All other nodal voltages are known. Thus, we need two nodal equations:

Applying KCL at node a, we have:

$$\frac{V_a - V_i}{R_1} + \frac{V_a - V_b}{R_2} + \frac{V_a - KV_b}{1/C_1 s} = 0$$

$$\Rightarrow \left(\frac{1}{R_1} + \frac{1}{R_2} + C_1 s\right)V_a - \left(\frac{1}{R_2} + KC_1 s\right)V_b = \frac{1}{R_1}V_1 \qquad (1)$$

Applying KCL at node b, we have:

$$\frac{V_b - V_a}{R_2} + C_2 s V_b = 0$$

$$\Rightarrow -\frac{1}{R_2}V_a + \left(\frac{1}{R_2} + KC_1 s\right)V_b = 0 \qquad (2)$$

Combining equations (1) and (2) by eliminating V_a, we get:

$$\frac{V_b}{V_i} = \frac{1}{R_1 R_2 C_1 C_2 s^2 + \left[R_2 C_2 + R_1 C_2 + R_1 C_1 (1-K)\right]s + 1}$$

Since $V_o = KV_b$, we have

$$\frac{V_o}{V_i} = \frac{K}{R_1 R_2 C_1 C_2 s^2 + \left[R_2 C_2 + R_1 C_2 + R_1 C_1 (1-K)\right]s + 1}$$

where $K = 1 + \frac{R_b}{R_a}$