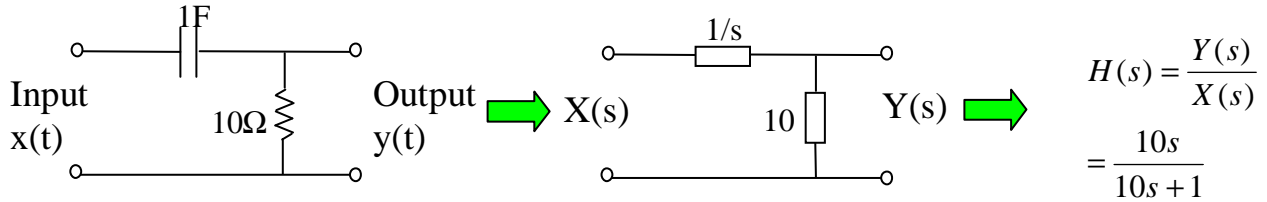


Building System with Laplace Transform

Example 1: A High-Pass Filter with zero initial condition



For any input $X(s)$, we can compute $Y(s)=H(s)X(s)$. We call this output response the **Zero-State Response**. In another words, $H(s)$ describes the functionality of this high-pass filter.

This concept applies to more than just circuit:

Example 2: Armature- Controlled dc servomotor (Example 6-15 in text)

Input : E_a (armature voltage)

Output : θ (angular shift)



$$H(s) = \frac{\theta(s)}{E_a(s)} = \frac{K_m}{s[(sL + R)(sJ + B) + K_m K_B]}$$

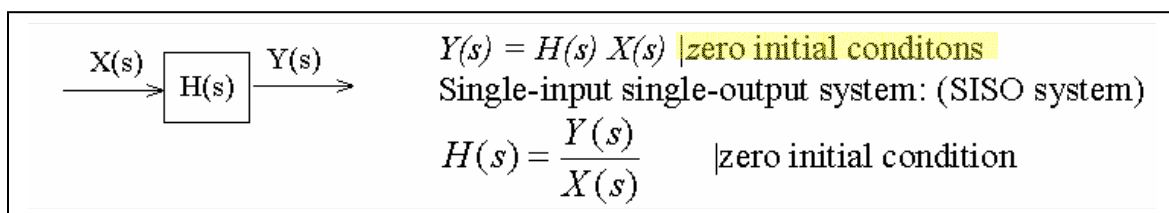
Key questions to ask:

1. How to find and understand $H(s)$?
2. How to choose appropriate components so that the system is stable?
3. How to compute the frequency response (~spectrum analyzer)?
4. How to combine multiple systems together?

Definition of a Transfer Function

Quantitative Description on how the system processes the input to form the output:

Transfer Function $H(s)$, in Laplace transform



Later we will show how initial conditions can be introduced.

All linear systems can be described by an ODE:

$$a_n \frac{d^{(n)} y(t)}{dt^n} + a_{n-1} \frac{d^{(n-1)} y(t)}{dt^{n-1}} + \dots + a_0 y(t) = b_m \frac{d^{(m)} x(t)}{dt^m} + b_{m-1} \frac{d^{(m-1)} x(t)}{dt^{m-1}} + \dots + b_0 x(t)$$

Taking the Laplace transform and **assume zero initial conditions**:

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \dots + a_0 Y(s) = b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \dots + b_0 X(s)$$

$$Y(S) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} X(s) \quad \text{or}$$

$$Y(s) = H(s) X(s) \quad \text{where} \quad H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

Remember product in s-domain corresponds to convolution in time-domain. Thus the differential equation can also be written in time-domain as a linear filtering!

$$y(t) = \int_0^t x(\tau) h(t - \tau) d\tau, \quad \text{where } h(t) = L^{-1}(H(s)) \text{ is the } \text{system impulse response}.$$

Non-zero initial conditions

This again:

$$(*) : a_n \frac{d^{(n)}y(t)}{dt^n} + a_{n-1} \frac{d^{(n-1)}y(t)}{dt^{n-1}} + \dots + a_0 y(t) = b_m \frac{d^{(m)}x(t)}{dt^m} + b_{m-1} \frac{d^{(m-1)}x(t)}{dt^{m-1}} + \dots + b_0 x(t)$$

Assume the initial state of the INPUT is zero as we have full control over it:

$$x(0) = x^{(1)}(0) = \dots = x^{(m-1)}(0) = 0$$

In many occasions, the system's initial states may not necessarily be zero.

It is easy in the Laplace transform to incorporate initial conditions. Recall from the differentiation theorem, we have

$$\begin{aligned} L\left[\frac{d^n y}{dt^n}\right] &= s^n Y(s) - s^{n-1}y(0^-) - s^{n-2}y^{(1)}(0^-) - \dots - sy^{(n-2)}(0^-) - y^{(n-1)}(0^-) \\ &= s^n Y(s) - (n-1)^{th} \text{ order polynomial based on initial conditions} \end{aligned}$$

where $y^{(i)}(0^-)$ denotes the i -th order time derivature of y evaluated at $t=0^-$.

Applying this rule to our differential equation (*) and grouping all the terms based on the initial conditions together:

$$\begin{aligned} &a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \dots + a_0 Y(s) - \text{polynomial based on initial conditions} \\ &= b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \dots + b_0 X(s) \end{aligned}$$

Let $C(s) =$ polynomial based on the initial condition. What is the degree of $C(s)$?

$$\begin{aligned} Y(s) &= \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} X(s) + \frac{C(s)}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} \\ &= H(s)X(s) + \frac{C(s)}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} \\ &= \text{Zero - state response} + \text{Zero - input response} \end{aligned}$$

Thus, there are **TWO COMPONENTS** to the output response

1. **Zero-State Response (ZSR)** or $H(s)X(s)$
 - It is called ZSR because it is the output response of the system if the initial state (condition) is zero.
2. **Zero-Input Response (ZIR)** or $\frac{C(s)}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$
 - It is called ZIR because it is the output response of the system if the input $X(s)$ is zero.
 - Notice that it has the same denominator as the transfer function, implying that its ROC is the same as the transfer function. We are going to use this fact later.

Separating the output response into two components that are based solely on input and initial conditions allow us to infer unknown initial conditions given a specific input. We will see an example in the homework.

Do not confuse ZSR and ZIR with another type of categorization of output response:

1. **Transient Response** – The part of the output response that approaches 0 as $t \rightarrow \infty$
2. **Forced (or Steady-State) Response** - what remains in the output as $t \rightarrow \infty$

Let's illustrate their differences with the following example:

Example 6-7

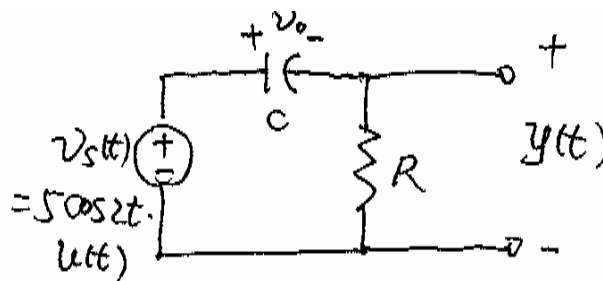
1st-order High Pass Filter

Input $v_s(t) = 5 \cos(2t)u(t)$

Output $y(t) = v_R(t)$

Initial capacitor voltage: $v_0 = -1$

$RC = 1$ second



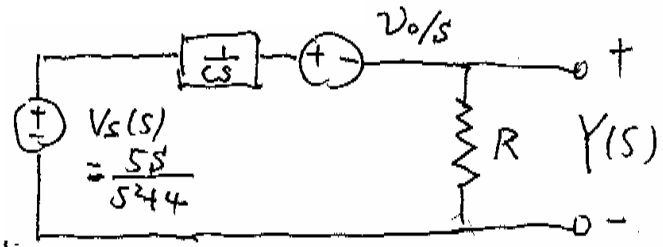
Find total response

$$V_s(s) = \frac{Y(s)}{R} \frac{1}{Cs} + \frac{v_0}{s} + Y(s)$$

$$Y(s) \left(\frac{1}{RCs} + 1 \right) = V_s(s) - \frac{v_0}{s}$$

$$Y(s) \left(\frac{1+s}{s} \right) = V_s(s) - \frac{v_0}{s}$$

$$Y(s) = \frac{s}{s+1} V_s(s) - \frac{v_0}{s+1}$$



It is clear from the first term is ZSR and the second is ZIR.

Note that I didn't substitute $v_0 = -1$ and $V_s(s) = 5s/(s^2 + 4)$ in the very beginning of the calculation. If I do that, I will obtain the output

$$Y(s) = \frac{4(s^2 - 1)}{(s+1)(s^2 + 4)}$$

and I would have no way of differentiating the two components.

(1) Find zero-input response and zero-state response

$$\text{Zero-input response: } L^{-1} \left[-\frac{v_0}{s+1} \right] = -v_0 e^{-t} u(t) = e^{-t} u(t)$$

$$\text{Zero-state response: } L^{-1} \left[\frac{s}{s+1} V_s(s) \right] = L^{-1} \left[\frac{s}{s+1} \cdot \frac{5s}{s^2 + 4} \right]$$

Apply partial fraction expansion, we got

$$\frac{s}{s+1} \cdot \frac{5s}{s^2 + 4} = \frac{1}{s+1} - \frac{-2-j}{s-j2} - \frac{-2+j}{s+j2} = \frac{1}{s+1} + \frac{2s-2}{s^2 + 4}$$

$$L^{-1} \left[\frac{s}{s+1} \cdot \frac{5s}{s^2 + 4} \right] = [e^{-t} + 4 \cos 2t - 2 \sin 2t] u(t)$$

(2) Find transient and forced response

$$\text{Combining ZIR and ZSR: } y(t) = [2e^{-t} + 4(\cos 2t) - 2(\sin 2t)] u(t)$$

Transient response: Which terms go to zero as $t \rightarrow \infty$?

Steady-state response: Which terms do not go to zero as $t \rightarrow \infty$?

Answer: Transient response = $2e^{-t} u(t)$

$$\text{Steady-state response} = [4 \cos(2t) - 2 \sin(2t)] u(t)$$

They are clearly different from ZSR and ZIR.