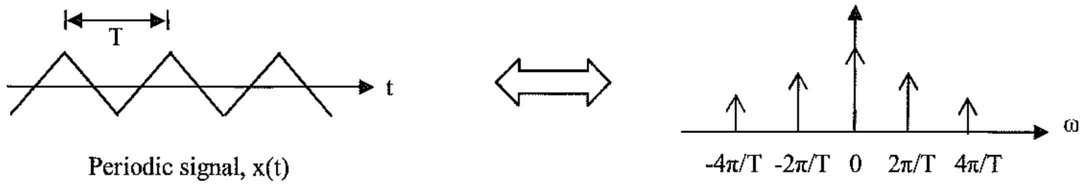
Chapter 10: Discrete Fourier Transform & Fast Fourier Transform

An assortment of "Fourier" analysis methods:

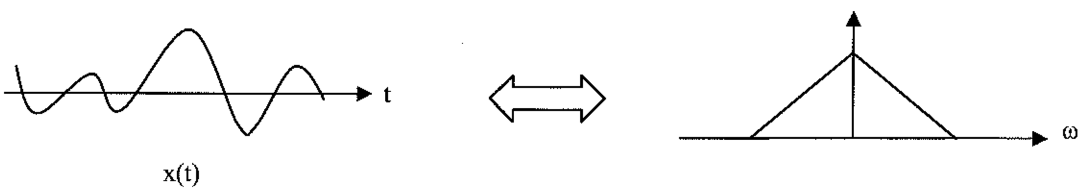
1. Fourier Series – continuous-time periodic signals



$$\textbf{Analysis} \ : X_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\frac{2\pi}{T}t}\,dt$$

$$\textbf{Synthesis} \ : x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\frac{2\pi}{T}t}$$

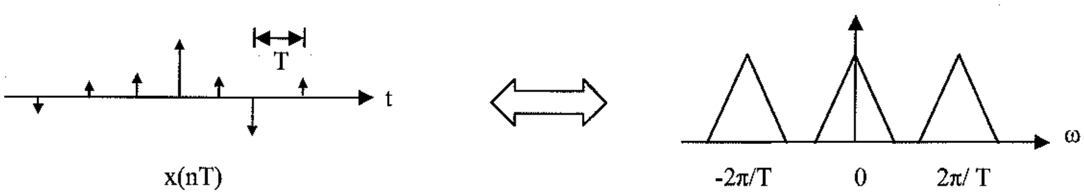
2. Fourier Transform – general continuous-time signals



$$\text{Analysis} : X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\text{Synthesis} : x(t) = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

3. Discrete-time Fourier Transform – general discrete-time signals

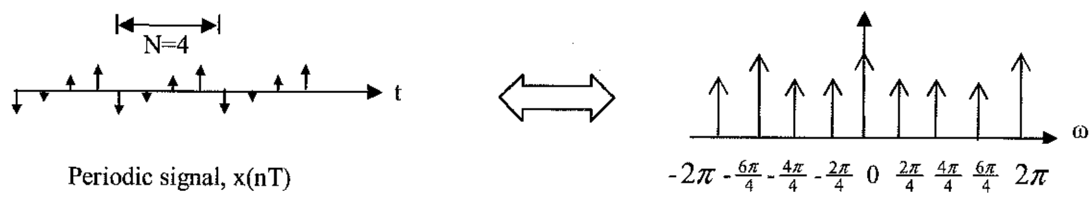


$$\text{Analysis} : X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}$$

$$\text{Synthesis} : x(nT) = \int_{-\infty}^{\infty} X(e^{j\omega T})e^{jn\omega T}\,d\omega$$

Signals	**Fourier**	**Transform Characteristics**
Continuous in t & Periodic	Fourier Series	Discrete in ω
Continuous in t	Continuous-time Fourier Transform	Continuous in ω
Discrete in t	Discrete-time Fourier Transform	Continuous in ω & Periodic
Discrete in t & Periodic	Discrete Fourier Transform	Discrete in ω & Periodic

Now we introduce the fourth one, the Discrete Fourier Transform (DFT).



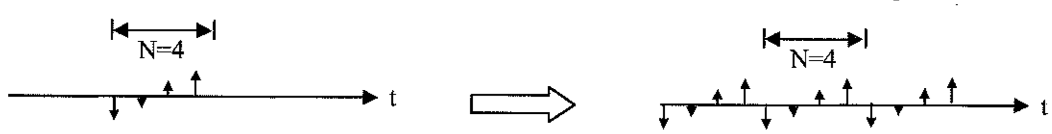
Periodic signal, x(nT)

$$\text{Analysis}: X_k = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi k}{N}n}, \quad k = 0,1,..,N-1$$

$$\text{Synthesis}: x(nT) = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi k}{N}n}, \quad n = 0,1,...,N-1$$

Why DFT? FOR "APPROXIMATING" DTFT OF A FINITE-DURATION SIGNAL

In real-life computations, ALL SIGNALS ARE FINITE – the consequence is that you can apply periodic extension to create a periodic discrete-time signal:



Finite-duration signal, x(nT)

Periodic signal, x'(nT)

What is the relation between the DTFT of the finite duration signal x(nT) and the DFT of the periodic extended signal x'(nT)?

DFT of x'(nT) are in fact FREQUENCY SAMPLES of the DTFT of x(nT)

Interpretation: Discrete-time Fourier series

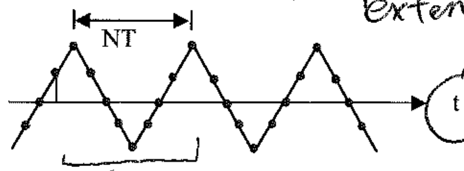
Recall the continuous-time Fourier Series for a periodic signal x(t) with period NT:

$$\text{Analysis}: \widetilde{X}_k = \frac{1}{NT}\int_0^{NT} x(t)e^{-jk\frac{2\pi}{NT}t}\,dt$$

$$\text{Synthesis}: x(t) = \sum_{k=-\infty}^{\infty} \widetilde{X}_k e^{jk\frac{2\pi}{NT}t}$$

Now, we sample x(t) with sampling period T, the resulting discrete-time sequence x(nT) is also periodic with period = N:

periodic extension ← → periodic extension



Time Domain

Original N samples

Now, we consider the continuous-time surrogate $x_s(t)$ of x(nT).

$$x_s(t) = \sum_{n=0}^{\infty} x(nT)\delta(t-nT)$$

This is also a periodic signal with period NT, so we compute its Fourier Series coefficient by integrating the product of one single period with exp(-j2πk/(NT)):

$$\left.\begin{aligned}\widehat{X}_k &= \frac{1}{NT}\int_0^{NT}\sum_{n=0}^{N-1} x(nT)\delta(t-nT)e^{-j\frac{2\pi k}{NT}t}\,dt\\ &= \frac{1}{NT}\sum_{n=0}^{N-1} x(nT)e^{-j\frac{2\pi k}{N}n}\end{aligned}\right\}\ \text{Fourier series coefficient}$$

If we define $X_k = NT\hat{X}_k$, then it coincides with the analysis formula of DFT:

$$X_k = \sum_{n=0}^{N-1} x(nT)e^{-j\frac{2\pi k}{N}n}$$

SAME AS

DFT

(Also known as Discrete Fourier Series)

DFT = Fourier ~~se~~ series cofficient of the periodic extension of the original finite-duration sequence.

Here is the definition of *Discrete Fourier Transform (DFT)* again:

$$\text{Analysis}: X_k = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi k}{N}n}, \quad k = 0,1,..,N-1$$

$$\text{Synthesis}: x(nT) = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi k}{N}n}, \quad n = 0,1,...,N-1$$

Compared with the analysis formula of the *DTFT of a finite-duration x(n)*:

$$X(e^{j\omega T}) = X(z)\Big|_{z=e^{j\omega T}} \qquad X(e^{j\omega T}) = \sum_{n=0}^{N-1} x(nT)e^{-jn\omega T} \leftarrow \text{DTFT}$$

continuous in ω

It is easy to see that X_k equals $X(e^{j\omega T})$ evaluated at $\omega = \frac{2\pi}{T}\frac{k}{N}$

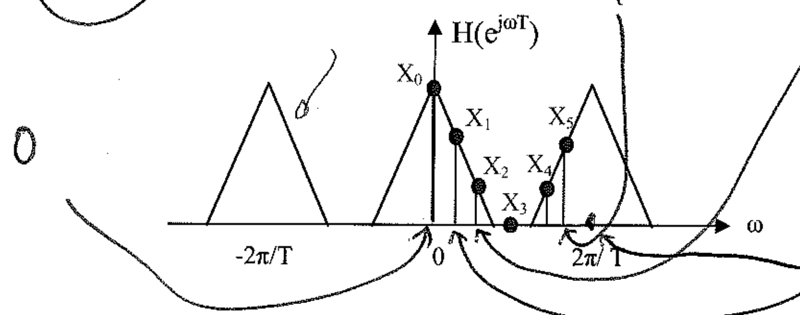
Question 1: What does it mean?

Recall that $X(e^{j\omega T})$ is periodic with period equal to the sampling frequency f_s or $2\pi/T$, thus X_k represents the k-th sample when sampling $X(e^{j\omega T})$ with N samples per period.

$\boxed{N=6}$

For example: if N=6, we have

$\omega_5 = \frac{2\pi(5)}{TN}$ $\quad \omega_2 = \frac{2\pi(2)}{TN}$ $\quad \omega_6 = \frac{2\pi(6)}{TN} = \frac{2\pi(6)}{T6} = \frac{2\pi}{T}$



Question 2: Is there any loss in information (do we need something similar to the Nyquist theorem in Frequency domain)?

No, as long as the number of samples per period is the same as or larger than the duration of the signal.

Interpretation
DFT = sampling in frequency of DTFT

Why? X_k is also the coefficient for the Discrete-time Fourier series of the periodic signal x'(nT).

** If the original x(nT) is finite length (N) then the DTFT is completely specified by these N frequency samples !!

Plugging it back to the Fourier Series synthesis formula, we have

$$x_s(t) = \frac{1}{N}\sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi k}{NT}t} \Rightarrow x(nT) = \frac{1}{N}\sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi k}{N}n}$$

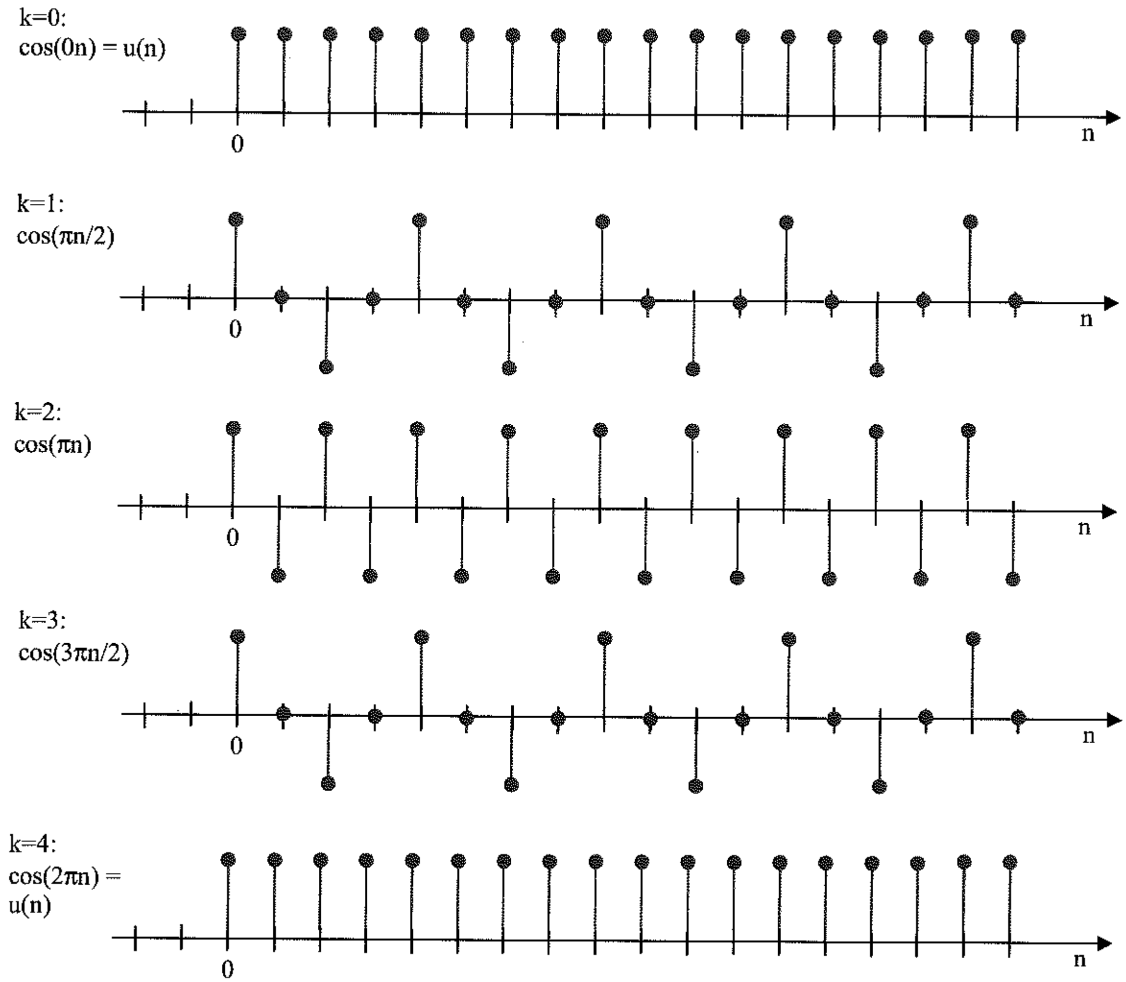
This is almost the same as the synthesis formula for DFT:

$$x(nT) = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi k}{N}n}, \quad n = 0,1,\ldots,N-1$$

Unlike continuous-time Fourier series which needs infinitely many harmonics, the discrete-time needs only N of them. The reason is that the discrete-time exponentials are themselves periodic in N as well:

$$\exp\left(j\frac{2\pi(k+N)}{N}n\right) = \exp\left(j\frac{2\pi k}{N}n + j2\pi n\right) = \exp\left(j\frac{2\pi k}{N}n\right)$$

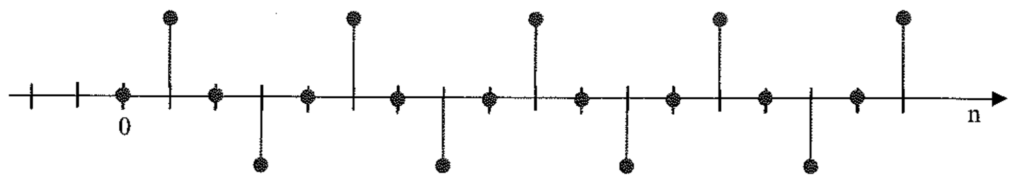
Let's consider the case when N=4. The real part: $\mathrm{Real}\left(\exp\left(j\frac{2\pi k}{4}n\right)\right) = \cos\left(\frac{2\pi k}{4}n\right)$:



The imaginary part : $\mathrm{Im}\left(\exp\left(j\frac{2\pi k}{4}n\right)\right) = \sin\left(\frac{2\pi k}{4}n\right)$

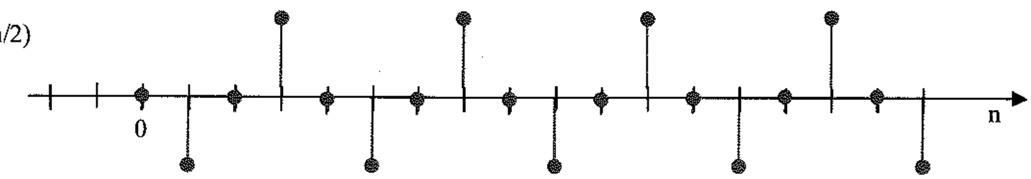
k=0:
sin(0n) = u(n)

k=1:
sin(πn/2)

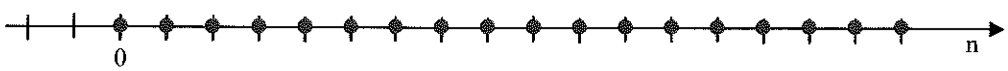


k=2:
sin(πn)

k=3:
sin(3πn/2)



k=4:
sin(2πn) = u(n)



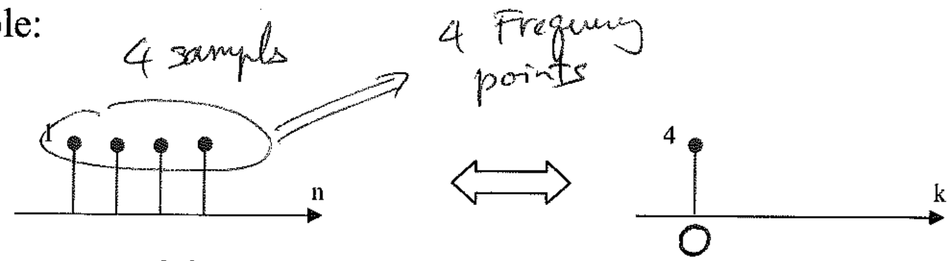
As you can see, $\exp\left(\frac{2\pi k}{4}n\right)$ has four distinct sequences before it starts repeating:

$$\exp\left(\frac{2\pi 0}{4}n\right) = u(n), \exp\left(\frac{2\pi}{4}n\right) = \exp\left(\frac{\pi}{2}n\right), \ \exp\left(\frac{2\pi 2}{4}n\right) = \exp(\pi n), \exp\left(\frac{2\pi 3}{4}n\right) = \exp\left(\frac{3\pi}{2}n\right)$$

DFT Theorem states that ANY periodic sequence x(n) with period = 4 can be written as a linear summation of these four sequences – the coefficients are the DFT of x(n):

$$x(n) = X_0 u(n) + X_1 \exp\left(\frac{\pi}{2}n\right) + X_2 \exp(\pi n) + X_3 \exp\left(\frac{3\pi}{2}n\right)$$

Example:



$$X_0 = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi 0}{4}n} = 1+1+1+1 = 4$$

$$X_1 = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi 1}{4}n} = e^{-j\frac{2\pi}{4}0} + e^{-j\frac{2\pi}{4}1} + e^{-j\frac{2\pi}{4}2} + e^{-j\frac{2\pi}{4}3} = 1 - j - 1 + j = 0$$

$$X_2 = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi 2}{4}n} = e^{-j\frac{4\pi}{4}0} + e^{-j\frac{4\pi}{4}1} + e^{-j\frac{4\pi}{4}2} + e^{-j\frac{4\pi}{4}3} = 1 - 1 + 1 - 1 = 0$$

$$X_3 = \sum_{n=0}^{N-1} x(nT) e^{-j\frac{2\pi 3}{4}n} = e^{-j\frac{6\pi}{4}0} + e^{-j\frac{6\pi}{4}1} + e^{-j\frac{6\pi}{4}2} + e^{-j\frac{6\pi}{4}3} = 1 + j - 1 - j = 0$$

$e^{-j\underbrace{\frac{2\pi}{4}\cdot n}_{\text{phase}}}$

phase = dividing a pizza into 4 slices you take n slices

Properties of the DFT

1. Linearity:

$$Ax(n) + By(n) \leftrightarrow AX(k) + BX(k)$$

2. Time Shift:

$$x(n-m) \leftrightarrow X(k)e^{-j2\pi km/N} = X(k)W_N^{k-m}$$

3. Frequency Shift:

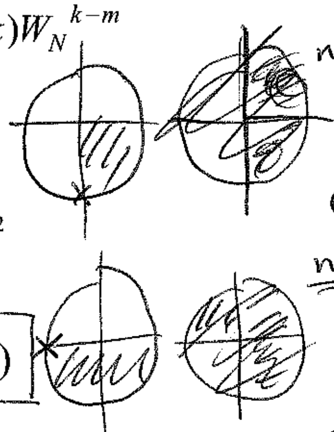
$$x(n)e^{j2\pi mn/N} \leftrightarrow X(k-m)$$

4. Parseval's Theorem

$$\sum_{n=0}^{N-1} |x(n)|^2 = N^{-1} \sum_{k=0}^{N-1} |X(k)|^2$$

* 5. Circular convolution

$$\boxed{x(n) \otimes y(n) \leftrightarrow X(k)Y(k)}$$



n=1

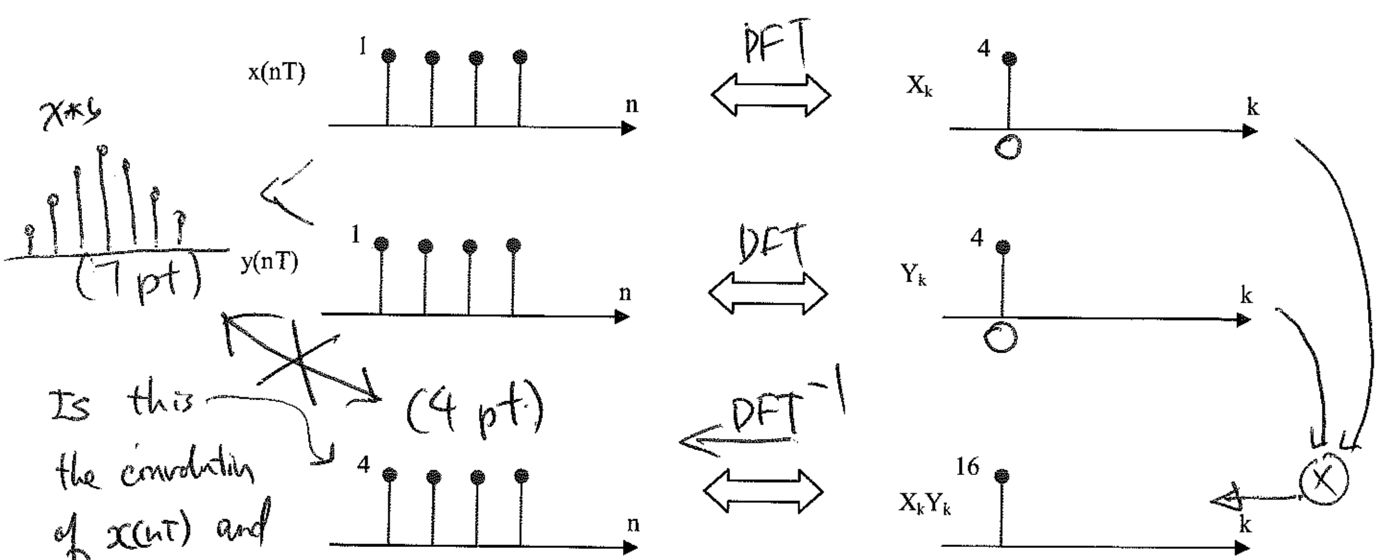
$\Rightarrow -\frac{\pi}{2}$

$e^{-j\frac{2\pi}{4}(1)} = -j$

n=2

$\Rightarrow -\pi$

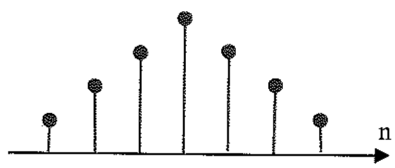
$e^{-j\frac{2\pi}{4}(2)} = -1$



Is this the convolution of x(nT) and y(nT)?

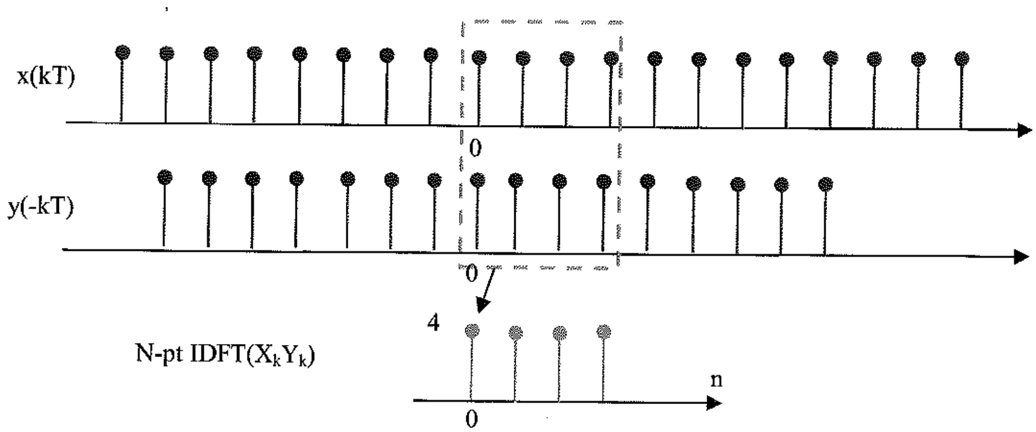
But convolving x(n) and y(n) should result in

x(nT) * y(nT)



You may recall that convolving two finite sequences of length N will result in a sequence of length 2N-1. Thus, it is OBVIOUSLY WRONG TO EXPECT THAT MULTIPLYING THEIR N-POINT DFT'S WILL GIVE THE RIGHT ANSWER.

When multiplying the two N-point DFT, an operation called circular convolution occurs which is equivalent to computing a N-point summation on the periodic extensions of the two original sequences:

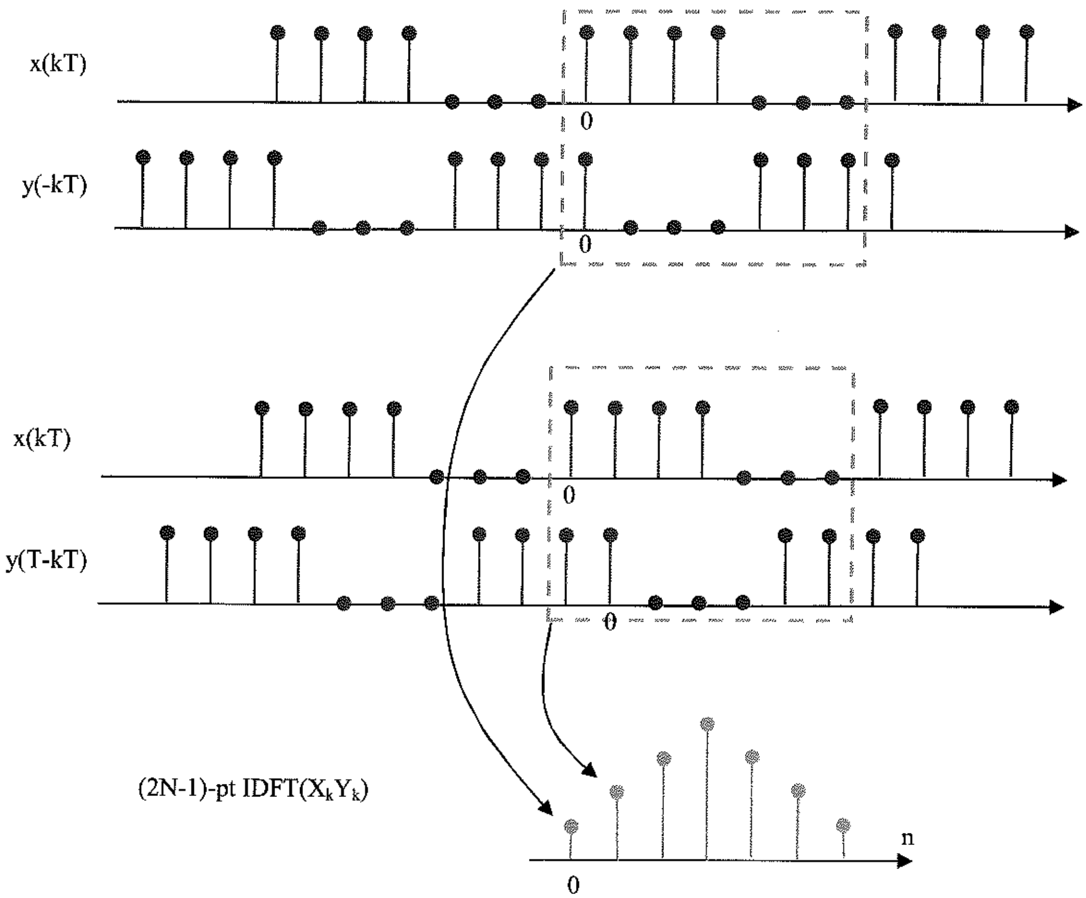


CIRCULAR CONVOLUTION

Multiplication of 2 N-pt DFT ⇒ time convolution of the periodic extended versions of the inputs ⇒ output is also periodic with period N

So what should you do in to order to compute x(n)*y(n) in the frequency domain?

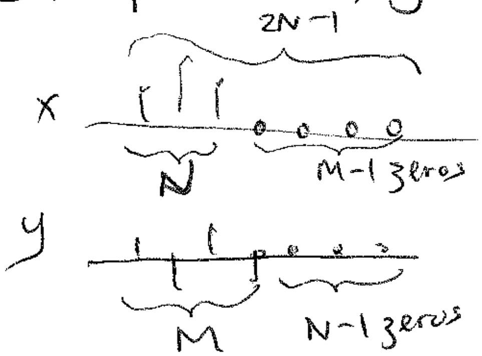
Answer: Zero-pad the original signal to 2N-1 and take a 2N-1 DFT



$x(n)$ N pt $\qquad$ $x(n) * y(n)$ ~~2N-1~~ pts N+M-1

$y(n)$ M pt

(1) Zero pad $x(n)$, $y(n)$ to ~~2N-1~~ N+M-1 pts



x ⇒ N+M-1-pt DFT X_k

y ⇒ N+M-1 pt DFT Y_k

$x(n) * y(n)$ ← N+M-1 pt DFT^{-1} ← $X_k \cdot Y_k$