5-3. Some Laplace Transform theorems

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Using these simple properties

- We can compute Laplace Transforms of a large number of functions based the
- Laplace transforms of \( \delta(t) \), \( u(t) \), and \( e^{-\alpha t} \) that we have computed.

1. To apply Laplace transform to real circuit problems (next chapter).

You are expected to

1. To be familiar with all the listed properties and transforms.
2. To understand the proofs (but I will not ask you to do any proof in tests.)
1. **Linearity**

\[ x(t) = a_1 x_1(t) + a_2 x_2(t) + \ldots + a_N x_N(t) \]

\[ X(s) = a_1 X_1(s) + a_2 X_2(s) + \ldots + a_N X_N(s) \]

Assume \( x(t) = a_1 x_1(t) + a_2 x_2(t) \) (\( a_1 \) and \( a_2 \) are time independent)

then

\[ X(s) = L[x(t)] = a_1 X_1(s) + a_2 X_2(s) \]

**Proof:**

\[ L[a_1 x_1(t) + a_2 x_2(t)] = \int_0^\infty (a_1 x_1(t) + a_2 x_2(t)) e^{-st} dt \]

\[ = a_1 \int_0^\infty x_1(t) e^{-st} dt + a_2 \int_0^\infty x_2(t) e^{-st} dt \]

\[ = a_1 X_1(s) + a_2 X_2(s) \]

(1) Find \( L(\cos \omega_0 t) \)

Key to solution: express \( \cos \omega_0 t \) as linear combination of \( \delta(t) \), \( u(t) \), and \( e^{-\alpha t} \). Note that

\[ e^{-j\omega_0 t} = \cos(-\omega_0 t) + j \sin(-\omega_0 t) \]

\[ = \cos(\omega_0 t) - j \sin(\omega_0 t) \]

\[ e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t) \]

\[ \Rightarrow \cos(\omega_0 t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \]

Since \( L[e^{-\alpha t}] = \frac{1}{s + \alpha} \), \( \text{Re}(s) > -\text{Re}(\alpha) \) \( \Rightarrow \)

\[ L[e^{-j\omega_0 t}] = \frac{1}{s + j\omega_0} \]

\[ L[e^{j\omega_0 t}] = \frac{1}{s - j\omega_0} \]

\[ \Rightarrow L[\cos(\omega_0 t)] = \frac{1}{2} [L(e^{j\omega_0 t}) + L(e^{-j\omega_0 t})] \]

**Q1:** ROC of \( L[\cos \omega_0 t] \)?

**Ans:** \( \text{Re}(s) > 0 \)

**Q2:** ROC of \( L[a x(t) + b x(t)] \)

**Ans:** ROC of \( L[x(t)] \) \( \cap \) ROC of \( L[x(t)] \) \( = \frac{s}{s^2 + \omega_0^2} \)

What is its region of convergence?

**Ans:** Intersection of all ROCs, and in this case, \( \text{Re}(s) > 0 \)

(2) Show that \( L[\sin(\omega t)u(t)] = \frac{\omega}{s^2 + \omega^2} \)

**proof:**

\[ \sin(\omega t) = \frac{1}{j} [e^{j\omega t} - e^{-j\omega t}] \]

(3) Is \( L[a(t)x(t) + b(t)y(t)] = A(s) X(s) + B(s) Y(s) \) and why?

**No** \( a(t) \) and \( b(t) \) are not constants!!
2. Complex Frequency shift \((s\text{-shift})\) Theorem

Assume \[ y(t) = x(t)e^{-\alpha t} \]
\[ X(s) = \mathcal{L}[x(t)] \quad \quad Y(s) = \mathcal{L}[y(t)] \]
Then \[ Y(s) = X(s + \alpha) \]

Proof:
\[ Y(s) = \int_0^\infty y(t)e^{-st}dt = \int_0^\infty x(t)e^{-st}e^{-\alpha t}dt = X(s + \alpha) \quad \text{Done.} \]

Example 5-4 Find \[ x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[ \frac{s + 8}{s^2 + 6s + 13} \right] \]

Solution: By “completing square”:
\[ X(s) = \frac{s + 8}{s^2 + 6s + 13} = \frac{(s + 3)^2 + 4}{(s + 3)^2 + 2^2} \]
\[ x(t) = e^{3t}\cos 2t + \frac{5}{2} e^{-3t}\sin 2t \]

3. Time Delay Theorem

Assume \[ L[x(t)] = L[x(t)u(t)] = X(s) \]
Then \[ L[x(t-t_0)u(t-t_0)] = e^{-st_0}X(s) \quad (t_0 > 0) \]

Proof: HOMEWORK
Example: Square wave beginning at \( t = 0 \)

\[
x_{sq}(t) = \mu(t) - 2\mu(t - \frac{T_0}{2}) + 2\mu(t - T_0)
\]

\[
x_{sq}(t) = -2\mu(t - \frac{3T_0}{2}) + 2\mu(t - 2T_0)
\]

\[
\vdots
\]

\[
L[x_{sq}(t)] = \frac{1}{s} - 2\frac{1}{s} e^{-\frac{T_0}{2s}} + 2\frac{1}{s} e^{-\frac{3T_0}{2s}} - 2\frac{1}{s} e^{-\frac{3T_0}{2s}} + 2\frac{1}{s} e^{-2T_0s} - \ldots
\]

Converged if \(|\lambda| < 1\)

\[
L[x_{sq}(t)] = \frac{1}{s} + 2\frac{1}{s} \lambda + 2\frac{1}{s} \lambda^2 + 2\frac{1}{s} \lambda^3 + 2\frac{1}{s} \lambda^4 + \ldots
\]

\[
= \frac{1}{s} + \frac{2}{s} (\lambda + \lambda^2 + \lambda^3 + \ldots)
\]

\[
= \frac{1}{s} + \frac{2\lambda}{s(1-\lambda)} = \frac{1}{s} (1 + \lambda) = \frac{1}{s} (1 - e^{-\frac{T_0}{2s}})
\]

\[
\sum_{n=1}^{\infty} \lambda^n = ?
\]

ROC: \( \{s : |\lambda| = \left| \exp\left(-\frac{1}{2} T_0 s\right) \right| = \exp\left(-\frac{1}{2} \text{Re}(s)\right) < 1\} = \{s : \text{Re}(s) > 0\} \) as \( \exp(-\beta) < 1 \) if \( \beta > 0 \)

4. Scaling

Assume \( X(s) = L[x(t)] \) then \( L[x(at)] = \frac{1}{a} X\left(\frac{s}{a}\right) \)

Restriction: \( a > 0 \)

\( x(at) \leftarrow a \) times fast (if \( a > 1 \)) or slow (if \( a < 1 \)) as \( x(t) \)

Proof:

\[
L[x(at)] = \int_{0}^{\infty} x(at)e^{-st} dt
\]

\[
= \frac{1}{a} \int_{0}^{\infty} x(\tau)e^{-\frac{s}{a}\tau} d\tau, \quad \text{set } \tau = at \quad \Rightarrow \quad dt = \frac{1}{a} d\tau
\]

\[
= \frac{1}{a} X\left(\frac{s}{a}\right)
\]
5. **Time Differentiation**

Assume \( X(s) = L[x(t)] \)

Then \( L\left(\frac{dx(t)}{dt}\right) = sX(s) - x(0^-) \)

Proof:

(1) Definition

\[
L\left(\frac{dx(t)}{dt}\right) = \int_0^\infty \frac{dx(t)}{dt} e^{-st} \, dt = \int_0^\infty e^{-st} \, dx(t)
\]

(2) Integration by parts:

\[
\int u(t)v'(t) \, dt = u(t)v(t) \Big|_{t=a}^{t=b} - \int v(t)u'(t) \, dt
\]

Make the following substitution:

\[
x(t) \Rightarrow v(t)
\]
\[e^{-st} \Rightarrow u(t)
\]

For all \( s \) inside \( \text{ROC of } X(s) \)

\[
\int_0^\infty [x(t)e^{st}] \, dt < \infty
\]

If we consider \( \lim_{t \to \infty} |x(t)e^{st}| \) \( \text{ROC} = \{s \mid \text{Re}(s) > 0\} \)

In the ROC of \( X(s) \), we must have \( \lim_{t \to \infty} [e^{-st}x(t)] = 0 \) otherwise the Laplace integral which is the area under \( e^{-st}x(t) \) from 0 to \( t=\infty \), will become \( \infty \).

Notes

- \( x(0^-) \) = the value of \( x(t) \) when \( t \) approaches 0 from the negative side (this is the convention: recall the definition of \( L(\delta(t)) \).
- What is the Region of Convergence? Same as \( X(s) \).
- What about higher derivatives?

Thus, it is easy to show that

\[
L\left(\frac{d^n}{dt^n} x(t)\right) = s^n X(s) - s^{n-1}x(0^-) - s^{n-2}x^{(1)}(0^-) - \ldots - sx^{(n-2)}(0^-) - x^{(n-1)}(0^-)
\]

where \( x^{(k)}(0^-) \) is the \( k \)-th derivative of \( x(t) \) evaluated at 0.

Consequence: Combined with linearity, you can now convert any ordinary differential equation AND initial conditions into Laplace Domain.
Example: Solve the following differential equation

\[
\frac{d^2}{dt^2}x(t) + 6\frac{d}{dt}x(t) + 5x(t) = e^{-7t}u(t) \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad \left[\frac{d}{dt}x(t)\right]_{t=0} = 0
\]

Write the differential equation in Laplace Transform:

\[
\begin{align*}
L\left[\frac{d^2}{dt^2}x(t)\right] &= s^2X(s) - sx(0^-) - x^{(1)}(0^-) = s^2X(s) - s - 0 \\
L\left[\frac{d}{dt}x(t)\right] &= sX(s) - x(0^-) = sX(s) - 1 \\
L[x(t)] &= X(s) \\
L[e^{-7t}u(t)] &= \frac{1}{s + 7}
\end{align*}
\]

By linearity, we have

\[
\begin{align*}
\frac{s^2}{(s^2 + 6s + 5)}X(s) - \frac{6}{(s^2 + 6s + 5)}X(s) - \frac{5}{(s^2 + 6s + 5)} &= \frac{1}{s + 7} \\
X(s) &= \frac{s^2 + 13s + 43}{(s + 7)(s^2 + 6s + 5)}
\end{align*}
\]

We will learn how to do the inverse transform in the next lesson, in the meantime we will use matlab:

\[
\text{>> ilaplace(sym('(s^2+7*s+7)/(s+7)/(s^2+6*s+5)'))} \\
\text{ans =} \\
\frac{3}{8}\exp(-5*t) + \frac{1}{24}\exp(-t) + \frac{7}{12}\exp(-7*t)
\]

Important: the Laplace transform of any ODE is a RATIONAL FUNCTION in $s$ (a ratio of two polynomials).

\[
X(s) = \frac{N(s)}{D(s)}
\]

where $N(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$ and $D(s) = s^m + b_{m-1}s^{m-1} + \ldots + b_0$ are Polynomials in $s$. 