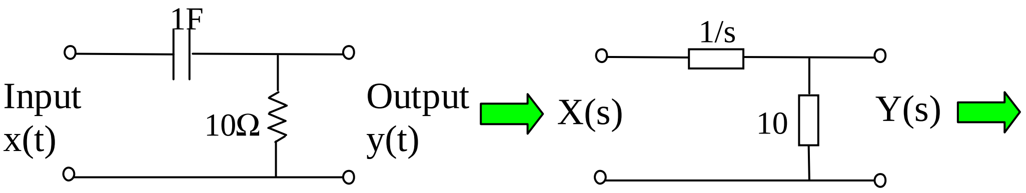
Building System with Laplace Transform

Convention in computing the transfer function:

Assume IC = 0

Example 1: A High-Pass Filter with zero initial condition



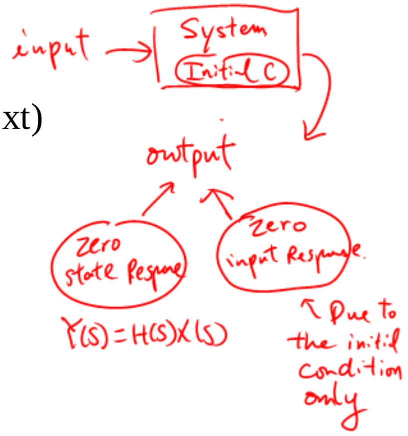
$$H(s)=\frac{Y(s)}{X(s)}$$

$$=\frac{10s}{10s+1}$$

For any input X(s), we can compute Y(s)=H(s)X(s). We call this output response the Zero-State Response. In another words, H(s) describes the functionality of this high-pass filter.

Contribution due to the input ALONE

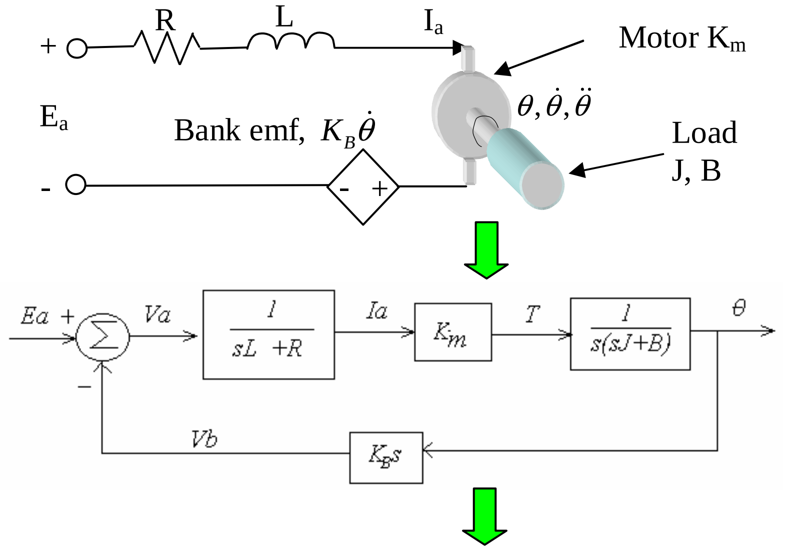
This concept applies to more than just circuit:



Example 2: Armature- Controlled dc servomotor (Example 6-15 in text)

Input : E_a (armature voltage)

Output : θ (angular shift)



$$H(s)=\frac{\theta(s)}{E_a(s)}=\frac{K_m}{s[(sL+R)(sJ+B)+K_mK_B]}$$

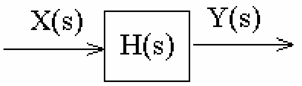
Key questions to ask:

1. How to find and understand H(s)?
2. How to choose appropriate components so that the system is stable?
3. How to compute the frequency response (~spectrum analyzer)?
4. How to combine multiple systems together?

Definition of a Transfer Function

Quantitative Description on how the system processes the input to form the output:

Transfer Function H(s), in Laplace transform



$Y(s) = H(s)\,X(s)$ |zero initial conditons

Single-input single-output system: (SISO system)

$$H(s) = \frac{Y(s)}{X(s)} \qquad |\text{zero initial condition}$$

Later we will show how initial conditions can be introduced.

All linear systems can be described by an ODE:

constant coeffcts

$$\overbrace{a_n\frac{d^{(n)}y(t)}{dt^n}+a_{n-1}\frac{d^{(n-1)}y(t)}{dt^{n-1}}+\ldots+a_0 y(t)}^{\text{output}}=b_m\frac{d^{(m)}x(t)}{dt^m}+b_{m-1}\frac{d^{(m-1)}x(t)}{dt^{m-1}}+\ldots+b_0 x(t)$$

$a_0\,y(t)$ $= b_m$

input

constant coeff

Taking the Laplace transform and assume zero initial conditions:

$$a_n s^n Y(s)+a_{n-1}s^{n-1}Y(s)+\ldots+a_0 Y(s)=b_m s^m X(s)+b_{m-1}s^{m-1}X(s)+\ldots+b_0 X(s)$$

$$Y(S)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_0}X(s) \quad \text{or}$$

$$Y(s)=H(s)X(s) \quad \text{where} \quad H(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_0}$$

ODE

Transfer function will always be a rational fn

Remember product in s-domain corresponds to convolution in time-domain. Thus the differential equation can also be written in time-domain as a linear filtering!

$y(t)=\int_0^t x(\tau)h(t-\tau)d\tau$, where $h(t)=L^{-1}(H(s))$ is the system impulse response.

Non-zero initial conditions

This again:

$$(*):\quad a_n\frac{d^{(n)}y(t)}{dt^n}+a_{n-1}\frac{d^{(n-1)}y(t)}{dt^{n-1}}+\ldots+a_0y(t)=b_m\frac{d^{(m)}x(t)}{dt^m}+b_{m-1}\frac{d^{(m-1)}x(t)}{dt^{m-1}}+\ldots+b_0x(t)$$

Assume the initial state of the INPUT is zero as we have full control over it:

$$x(0)=x^{(1)}(0)=\ldots=x^{(m-1)}(0)=0$$

In many occasions, the system's initial states may not necessarily be zero.
It is easy in the Laplace transform to incorporate initial conditions. Recall from the differentiation theorem, we have

$$\begin{aligned}L\left[\frac{d^ny}{dt^n}\right] &= s^nY(s)-\overbrace{s^{n-1}y(0^-)-s^{n-2}y^{(1)}(0^-)-\ldots-sy^{(n-2)}(0^-)-y^{(n-1)}(0^-)}^{\text{initial conditions}}\\ &= \underline{s^nY(s)}-\boxed{(n-1)^{th}\text{ order polynomial based on initial conditions}}\end{aligned}$$

where $y^{(i)}(0^-)$ denotes the i-th order time derivative of y evaluated at $t=0^-$.

Applying this rule to our differential equation (*) and grouping all the terms based on the initial conditions together:

$$\begin{aligned}&a_ns^nY(s)+a_{n-1}s^{n-1}Y(s)+\ldots+a_0Y(s)-\text{polynomial based on initial conditions}\\&=b_ms^mX(s)+b_{m-1}s^{m-1}X(s)+\ldots+b_0X(s)\end{aligned}$$

↑ What is the degree?
Ans: n-1

Let C(s) = polynomial based on the initial condition. What is the degree of C(s)?

$$\begin{aligned}Y(s) &= \frac{b_ms^m+b_{m-1}s^{m-1}+\ldots+b_0}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_0}X(s)+\frac{\text{C(s)}}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_0}\\ &= H(s)X(s)+\frac{\text{C(s)}}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_0}\\ &= \underline{\underline{\text{Zero-state response}}}+\underline{\underline{\text{Zero-input response}}}\end{aligned}$$

Thus, there are TWO COMPONENTS to the output response

1. *Zero-State Response (ZSR)* or $H(s)X(s)$
    - It is called ZSR because it is the output response of the system if the initial state (condition) is zero.
2. *Zero-Input Response (ZIR)* or $\dfrac{C(s)}{a_n s^n + a_{n-1} s^{n-1} + ... + a_0}$
    - It is called ZIR because it is the output response of the system if the input X(s) is zero.
    - Notice that it has the same denominator as the transfer function, implying that its ROC is the same as the transfer function. We are going to use this fact later.

↑ Provided there is no pole cancellation

Separating the output response into two components that are based solely on input and initial conditions allow us to infer unknown initial conditions given a specific input. We will see an example in the homework.

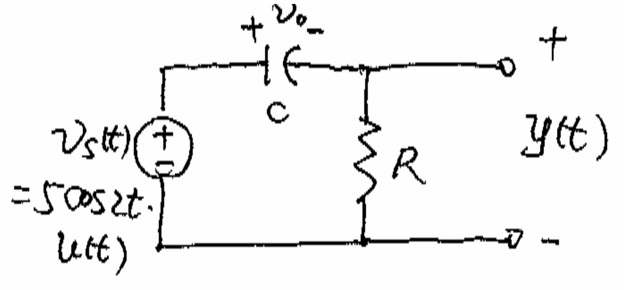
Do not confuse ZSR and ZIR with another type of categorization of output response:

1. *Transient Response* – The part of the output response that approaches 0 as t→∞
2. *Forced (or Steady-State) Response* - what remains in the output as t→∞

Let's illustrate their differences with the following example:

Example 6-7
1^{st}-order High Pass Filter

Input $v_s(t) = 5\cos(2t)u(t)$
Output $y(t) = v_R(t)$
Initial capacitor voltage: v_0=-1
RC = 1 second



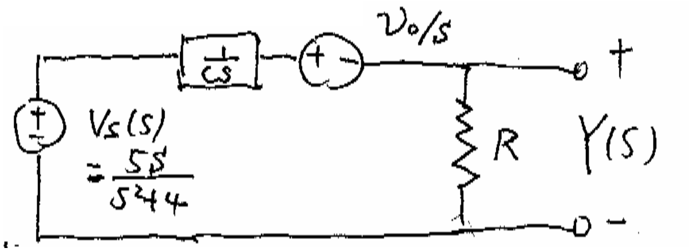
Find total response

$$V_S(s)=\frac{Y(s)}{R}\frac{1}{Cs}+\frac{v_0}{s}+Y(s)$$

$$Y(s)\left(\frac{1}{RCs}+1\right)=V_s(s)-\frac{v_0}{s}$$

$$Y(s)\left(\frac{1+s}{s}\right)=V_s(s)-\frac{v_0}{s}$$

$$Y(s)=\overbrace{\underbrace{\boxed{\frac{s}{s+1}}}_{H(s),\ \text{Transfer fn}}V_s(s)}^{\text{ZSR}}-\boxed{\frac{v_0}{s+1}}\rightarrow \text{ZIR}$$



It is clear from the first term is ZSR and the second is ZIR.
Note that I didn't subsitute v_0=-1 and $V_s(s)=5s/(s^2+4)$ in the very beginning of the calculation. If I do that, I will obtain the output

$$Y(s)=\frac{4(s^2-1)}{(s+1)(s^2+4)}$$

and I would have no way of differentiating the two components.

(1) Find zero-input response and zero-state response

Zero-input response: $L^{-1}\left[-\frac{v_0}{s+1}\right]=-v_0e^{-t}u(t)\ =e^{-t}u(t)$

Zero-state response: $L^{-1}\left[\frac{s}{s+1}V_s(s)\right]=L^{-1}\left[\frac{s}{s+1}\cdot\frac{5s}{s^2+4}\right]$

Apply partial fraction expansion, we got

$$\frac{s}{s+1}\cdot\frac{5s}{s^2+4}=\frac{1}{s+1}-\frac{-2-j}{s-j2}-\frac{-2+j}{s+j2}=\frac{1}{s+1}+\frac{2s-2}{s^2+4}$$

$$L^{-1}\left[\frac{s}{s+1}\cdot\frac{5s}{s^2+4}\right]=[e^{-t}+4\cos 2t-2\sin 2t]u(t)$$

(2) Find transient and forced response

Combining ZIR and ZSR: $y(t)=[2e^{-t}+4(\cos 2t)-2(\sin 2t)]u(t)$

Transient response: Which terms go to zero as t→∞?

Steady-state response: Which terms do not go to zero as t→∞?

Answer: Transient response = $2e^{-t}u(t)$

Steady-stte response = $[4\cos(2t)-2\sin(2t)]u(t)$

They are clearly different from ZSR and ZIR.