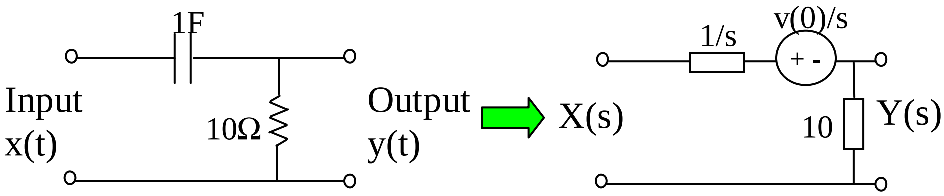
More about ZSR and ZIR

1. Finding unknown initial conditions:

Given the following circuit with unknown initial capacitor voltage v(0):



$$H(s)=\frac{Y(s)}{X(s)}=\frac{10s}{10s+1}$$

Simple mesh analysis yield

$$X(s)=\frac{Y(s)}{10s}+\frac{v(0)}{s}+Y(s)$$
$$Y(s)=\frac{10s}{10s+1}X(s)-\frac{10v(0)}{10s+1}$$

Immediately, we know that the transfer function H(s) is

$$H(s)=\frac{10s}{10s+1}$$

and the ZSR and ZIR are respectively

$$\checkmark\quad Y_{ZSR}(s)=\frac{10s}{10s+1}X(s)$$
$$Y_{ZIR}(s)=-\frac{10v(0)}{10s+1}$$

So, if you use a step function $x(t)=u(t)$ as input, you get an exponential decay output $y(t)=\frac{1}{2}e^{-\frac{1}{10}t}u(t)$ ← total response = $Y_{ZSR}+Y_{ZIR}$

$$Y(s)=\mathcal{L}(y(t))=\mathcal{L}\left[\frac{1}{2}e^{-\frac{1}{10}t}u(t)\right]=\frac{1/2}{s+1/10}=\frac{5}{10s+1}$$

$$X(s)=\mathcal{L}[x(t)]=\mathcal{L}[u(t)]=\frac{1}{s}$$

$$Y(s)=Y_{ZSR}(s)+Y_{ZIR}(s)=\frac{10s}{10s+1}\cdot\frac{1}{s}-\frac{10v(0)}{10s+1}$$
$$=\frac{10}{10s+1}-\frac{10v(0)}{10s+1}$$
$$=\frac{10[1-v(0)]}{10s+1}=Y(s)=\frac{5}{10s+1}$$

$$\Rightarrow 10[1-v(0)]=5\Rightarrow v(0)=\frac{1}{2}\,V$$

2. Find out about an unknown system:

$$X(s) \longrightarrow \boxed{H(s) \text{ and unknown IC}} \longrightarrow Y(s)$$

How do we find an unknown system and initial conditions?

Answer: Input a test signal and measure the output. The use the I/O relationship to figure out the system

Example: The output of the system is $Y(s) = \dfrac{5}{10s+1}$ for an input $X(s) = \dfrac{1}{s}$

So, is the transfer function $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{5s}{10s+1}$? → Did not take into the acount the INITIAL CONDITION! Remember H(s) assumes Zero IC!!

NO, we've forgotten the Initial Conditions!!
It should be clear that ONE pair of I/O signals will not be sufficient, we need two. Why? As we have learnt, the total output response of a system is

$$Y(s) = H(s)X(s) + \frac{C(s)}{D(s)}$$

If we have two I/O pairs, we have

$$Y_1(s) = H(s)X_1(s) + \frac{C(s)}{D(s)} \quad (1)$$

$$Y_2(s) = H(s)X_2(s) + \frac{C(s)}{D(s)} \quad (2)$$

Subtracting (2) from (1) eliminates the initial conditions and obtains the transfer function:

$$H(s) = \frac{Y_1(s) - Y_2(s)}{X_1(s) - X_2(s)} \quad \checkmark$$

The initial conditions can then be computed by substituting H(s) back to (1) or (2).

Try the above example again with another I/O pairs: $X(s) = 1$ and $Y(s) = \dfrac{10s+4}{10s+1}$

$X_1(s) = \frac{1}{s}$, $Y_1(s) = \frac{5}{10s+1}$

$X_2(s) = 1$, $Y_2(s) = \frac{10s+4}{10s+1}$

$$\Rightarrow H(s) = \frac{\frac{10s+4}{10s+1} - \frac{5}{10s+1}}{1 - \frac{1}{s}} = \frac{\frac{10s-1}{10s+1}}{\frac{s-1}{s}} = \frac{(10s-1)s}{(10s+1)(s-1)}$$

$$Y_2(s) = \frac{10s+4}{\boxed{10s+1}} = H(s)X_2(s) + \frac{C(s)}{D(s)} = \frac{(10s-1)s}{(10s+1)(s-1)} 1 + \frac{C(s)}{(10s+1)(s-1)}$$

$$= \frac{(10s-1)s + C(s)}{\boxed{10s+1}(s-1)}$$

$(10s-1)s + C(s) = (s-1)(10s+4)$

$\therefore C(s) = (s-1)(10s+4) - (10s-1)s$

Stability of systems

We first need to introduce the concepts of BOUNDED and TRANSIENT signals.
There are two types of well-behaved signals:

1) Bounded: $|x(t)| \le M < \infty$ for all t.
2) Transient: $\lim_{t\to\infty} x(t) = 0$

Are these signals bounded? Are they transient?

1) $\delta(t)$ Transient, Not bounded
2) $\cos(t)u(t)$ Not tran, Bounded
3) $t\sin(t)u(t)$ Not tran, Not Bounded
4) $e^{-t}u(t)$
5) $e^{t}u(t)$

Answers:

1) $\delta(t)$	Not bounded	and	transient
2) $\cos(t)u(t)$	Bounded	and	not transient
3) $t\sin(t)u(t)$	Not bounded	and	not transient
4) $e^{-t}u(t)$	Bounded	and	transient
5) $e^{t}u(t)$	Not bounded	and	not transient

Can you tell from their Laplace transforms?

Exclude the imaginary axis !!

> A sufficient condition for a signal to be a <u>transient signal</u>:
> The poles on its Laplace Transform must be on the open left half plane.

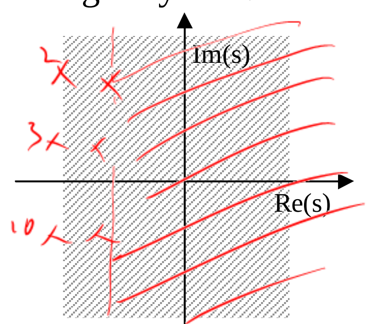
This condition is the same as saying the ROC contains the imaginary axis.

$\Rightarrow$ the ROC contains the origin

$\Rightarrow \int_0^\infty |x(t)|dt < \infty$

$\Rightarrow \lim_{t\to\infty} x(t) = 0$

$\Rightarrow$ transient



> A <u>bounded signal</u> must satisfy the following two conditions:
> 1. Its Laplace transform must be proper.
>
> i.e. if $X(s) = \frac{N_X(s)}{D_X(s)}$, degree($N_X(s)$)<degree($D_X(s)$)
>
> 2. The poles must either be
>
> a/ on the open left half plane or
>
> b/ on the imaginary axis AND simple (i.e. multiplicity = 1).

$\frac{1}{(s^2+1)} = \frac{1}{(s-j)(s+j)}$ ✓

$\frac{1}{(s^2+1)^2} = \frac{1}{(s-j)^2(s+j)^2}$ X

Reason for Condition 1:

Consider $\delta(t)$, $L[\delta(t)] = 1$ is not proper.

For general X(s), if degree($N_X(s)$)≥degree($D_X(s)$), applying long division:

$$X(s) = a_n s^n + a_{n-1}s^{n-1} + ... + a_0 + \frac{R_X(s)}{D_X(s)} \Rightarrow L^{-1}[X(s)] = a_n\delta^{(n)}(t) + +a_{n-1}\delta^{(n-1)}(t) + ... + a_0\delta(t) + ...$$

Thus, x(t) is not bounded due to the delta functions.

Reason for Condition 2:

Okay for the LHP poles to have multiplicity > 1 $\because$ exponential signal decay faster!!

If all the poles are on the open left half plane, we know the signal decays to zero.
If there are poles on the imaginary axis, there are two cases:

Case 1: Poles are simple, i.e. multiplicity = 1 such as L[cos(t)] = s/(s^2+1). In this case, the signal is oscillating but still bounded.

Case 2: Poles are not simple, i.e. multiplicity > 1 such as L[tsin(t)]=1/(s^2+1)^2. In this case, the signal is not bounded.

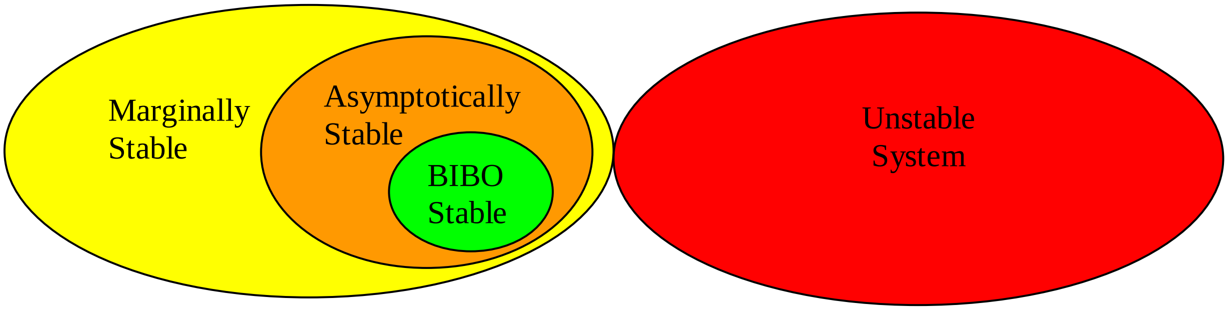
expanding envelope
⇒ unbounded

Now we come back to study the stability of a system.

All systems can be classified into stable or unstable. There are three different types of stable systems.

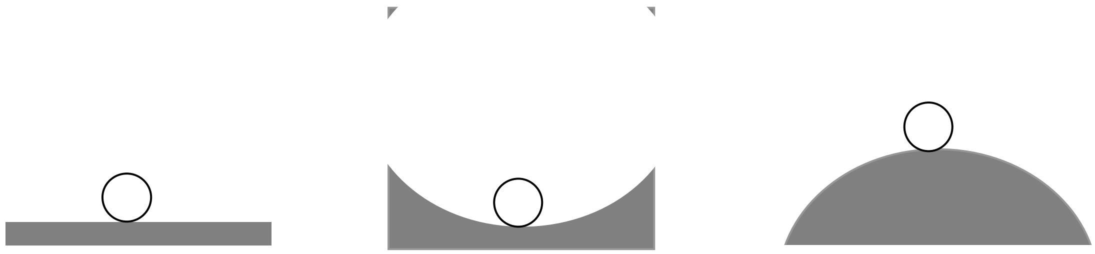
1. Bounded-Input Bounded-Output Stable (BIBO) System
2. Asymptotic Stable System
3. Marginally Stable System

Here is a Venn diagram that describes the classification.



BIBO describes systems behavior subjected to general input, assuming zero initial conditions.

Asymptotic, marginally, and unstable systems refer to the long-term behavior of a system under any initial conditions but no input.



Marginally Stable | Asymptotically Stable | Unstable

Recall: Given a system $H(s) = \frac{N(s)}{D(s)}$ and input $X(s)$, the most general form of output is $\underbrace{Y(s)}_{} = \underbrace{\frac{C(s)}{D(s)}}_{\text{ZIR}} + \underbrace{\overbrace{\frac{N(s)}{D(s)}}^{\text{Transfer fn}} \overbrace{X(s)}^{\text{Input}}}_{\text{ZSR}}$

The first term is the ZIR and the second term is ZSR.

<u>BIBO Stability</u>

Definition: Assume zero initial state, a system H(s) is BIBO stable if it outputs a bounded output for any bounded input.

> A BIBO system H(s) must satisfy the following two conditions:
> 1. If $H(s) = \frac{N(s)}{D(s)}$ degree($N(s)$) $\leq$ degree($D(s)$). (Not the same as proper ∵ proper ≜ deg(N(s)) < deg(D(s)))
> 2. The poles of H(s) must be on the open left half plane.

As there is zero initial condition, the output Y(s) is just $\frac{N(s)}{D(s)} X(s)$. To ensure Y(s) is bounded, we need to check two things:

① proper
② poles ∈ LHP or poles ∈ jw-axis and simple

$Y(s) = \frac{N(s)}{D(s)} \cdot X(s) = \frac{N(s)}{D(s)} \cdot \frac{X_N(s)}{X_D(s)}$ → deg(N(s)) + deg(X_N(s)) want to be bounded as well

Know that deg(X_N(s)) < deg(X_D(s))

deg(N(s)) ≤ deg(D(s)) ⇒ deg(N(s)) + deg(X_N(s)) < deg(D(s)) + deg(X_D(s))

⇒ Y(s) is proper.

1. degree[numerator of Y(s)] < degree[denominator of Y(s)]

   degree[numerator of Y(s)] = degree[N(s)] + degree[numerator of X(s)]
   degree[denominator of Y(s)] = degree[D(s)] + degree[denominator of X(s)]
   Since X(s) is bounded, degree[numerator of X(s)] < degree[denominator of X(s)], thus all we need is **degree[N(s)] ≤ degree[D(s)].**

2. Y(s) has either open left half plane poles and/or simple poles on jω-axis.

   As X(s) is bounded, its poles must either be on the open left half poles or simple on the imaginary axis. To ensure Y(s) = H(s)X(s) satisfied the same criteria, all the poles of H(s) must be on the open left half plane.

   H(s) cannot have any pole on the imaginary axis, not even simple one because the input X(s) might also have a simple pole at the same location and the resulting Y(s) will have DOUBLE imaginary poles, making it non-bounded.

Asymptotic Stable

Definition: Assume zero input, a system is asymptotic stable if it gives a transient output for ANY initial state.

A system is asymptotically stable if all the poles of the Laplace transform $\frac{1}{D(s)}$ are on the open left half plane.

This is easy. As there is no input, the output is $Y(s)=\frac{C(s)}{D(s)}$. In general, Y(s) will have the same poles as $\frac{1}{D(s)}$ unless there are cancellations of poles and zeros. Thus to ensure y(t) is transient, all we need is to ensure that $\frac{1}{D(s)}$ has open left half plane poles.
Also, it is easy to see that BIBO stability $\Rightarrow$ Asymptotic stability.

Marginally Stable

Definition: Assume zero input, a system is marginally stable if it gives a bounded output for ANY initial state.

> A system is marginally ~~asymptotically~~ stable if all the poles of the Laplace transform $\frac{1}{D(s)}$ are either on the open left half plane or simple on the imaginary axis.

To ensure $Y(s) = \frac{C(s)}{D(s)}$ is bounded, we need to check:

1. degree[C(s)] < degree[D(s)]. This is always true. See page 6-15.
2. Poles of Y(s) are either on the open left half plane or simple on the imaginary axis.

Also, it is easy to see that BIBO stability $\Rightarrow$ Asymptotic stability $\Rightarrow$ Marginally stability.

Unstable system

Definition: Assume zero input, a system is unstable if it gives a unbounded output for some initial state.
An unstable system is a system that is NOT marginally stable.

$$\text{BIBO} \Leftrightarrow \begin{cases} ① \ H(s) = \frac{N(s)}{D(s)} : \deg(N(s)) \le \deg(D(s)) \\ ② \ H(s) \text{ has only LHP poles} \end{cases}$$

Asymptotic $\Leftrightarrow$ $\frac{1}{D(s)}$ has only LHP poles

Marginal $\Leftrightarrow$ $\frac{1}{D(s)}$ has LHP poles or jw-axis & simple poles

If a system is BIBO, is it asymptotically stable? YES
is it marginally stable? YES

If a system is asym stable, is it BIBO? Maybe.
is it marginally stable? YES

If a system is marginally, is it BIBO? Maybe
is it asym stable? Maybe