6-5 Routh Array

Both asymptotically stability and BIBO stability require checing whether *H(s) = N(s)/D(s)* has all its poles on the open left half plane.

How to check?

Solution 1: Factorize D(s):

$$\begin{aligned} D(s) &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \\ &= (s-\alpha_1)(s-\alpha_2)\dots(s-\alpha_n) \\ stable &\Leftrightarrow all \quad \mathrm{Re}(\alpha_j) < 0 \end{aligned}$$

This is difficult because there are NO CLOSED FORM for polynomials of degree higher than four. Numerical methods are:

1. Computatinally intensive
2. Solution are not exact.

Solution 2: Routh Array (Routh-Hurwitz criterion). This is EASY!
Find how many poles in the right half of the s-plane?

(1) Basic Method

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \dots + a_0$$

Degree of the leading term

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & \\ s^{n-3} & c_1 & c_2 & c_3 & \\ \vdots & & & & \\ s^0 & q & & & \end{array}$$

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_3 = \frac{a_{n-1} \cdot a_{n-6} - a_n \cdot a_{n-7}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}$$

The Routh-Hurwitz Criterion: Number of sign changes in the first column of the array = number of poles in the (OPEN) r. h. p.

Example 6-8

$$\boxed{D(s)=s^3+14s^2+41s-56}$$

$$\begin{array}{c|cc}
s^3 & 1 & 41 \\
s^2 & 14 & -56 \\
s^1 & b_1 = 45 & b_2 = 0 \\
s^0 & c_1 = -56 & c_2 = 0
\end{array}$$

$$b_1=\frac{14\times 41-1\times(-56)}{14}=45$$

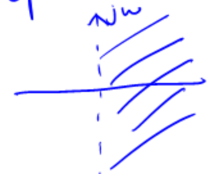
$$b_2=\frac{14\times 0-1\times 0}{14}=0$$

$$c_1=\frac{45\times(-56)-14\times 0}{45}=-56$$

1, 14, 45, -56

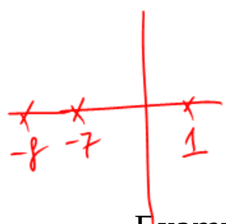
sign: Changed once => one pole in the r.h.p

open R.H.P



Does not include the jw axis

verification:



$$D(s)=s^3+14s^2+41s-56$$
$$=(s-1)(s+7)(s+8)$$

Example 6-9

$$D(s)=s^4+5s^3+s^2+10s+1$$

$$b_1=\frac{5\times1-1\times10}{5}=-1$$

$$b_2=\frac{5\times1-1\times0}{5}=1$$

$$\begin{array}{c|ccc}
s^4 & 1 & 1 & 1 \\
s^3 & 5 & 10 & 0 \\
s^2 & b_1=-1 & b_2=1 & 0 \\
s^1 & c_1=15 & c_2=0 & \\
s^0 & d_1=1 & &
\end{array}$$

$$c_1=\frac{-1\times10-5\times1}{-1}=15$$

$$c_2=\frac{-1\times0-5\times0}{-1}=0$$

$$d_1=\frac{15\times1-(-1)\times0}{15}=1$$

Sign: changed twice => two poles in r.h.p.

FAQ about Routh Array

1. Why do number of sign changes in Routh array has anything to do with the number of r.h.p. poles?
   Let say $D(s) = (s - p_1)(s - p_2)(s - p_3)$ p_1, p_2, p_3 are real
   Expand it out you get, $D(s) = s^3 - (p_1 + p_2 + p_3)s^2 + \ldots$
   In this case, the Routh array looks like this:

if $-(p_1+p_2+p_3)<0$ $\Rightarrow$ at least one of p_1, p_2, p_3 is positive!

s^3	1	...
s^2	$-(p_1 + p_2 + p_3)$	...
s	...	

INERTIA = (p, q)

p = # of poles on the r.h.p

q = # of poles on the l.h.p.

   If all the poles are on the left half plane, $-(p_1 + p_2 + p_3)$ must be positive and there will not be a sign change between the first and second row. If there is a sign change, we must have at least ONE pole on the open right half plane.

Routh array operations does not change the INERTIA of the original polynomial

2. What about the rest of the rows?
   A detail explanation of the Routh-Hurwitz criterion is beyond the scope of this course. An elementary proof of this criterion can be found in the paper titled "Elementary proof of the Routh-Hurwitz test" by G. Meinsma in Systems & control Letters 25 (1995) p. 237-242.

3. What about poles on the imaginary axis?
   It is in fact possible to also keep track of the number of pure-imaginary poles from the Routh array as well. However, we will not consider such procedure here.

4. What happen when there are zeros in the ~~first~~ second column?
   Excellent question! There are two cases

Case 1: Only the first element of the row is 0 but the rest of the row is neither entirely zero nor empty.

s^4	$\cancel{0}$ ε	2	1
s^3	$\cancel{3}$ ε only this entry		

Procedure:

1. replace 0 by a small ε
2. Find # sign changes in the first column for either ε>0 and ε<0. Both cases should give you the same result.

Example 6-10 $D(s) = s^4 + s^3 + s^2 + s + 3$

$$
\begin{array}{c|ccc}
s^4 & 1 & 1 & 3 \\
s^3 & 1 & 1 & \\
s^2 & b_1 = \varepsilon & b_2 = 3 & b_3 = 0 \\
s^1 & c_1 = \dfrac{\varepsilon - 3}{\varepsilon} & c_2 = 0 & \\
s^0 & d_1 = 3 & &
\end{array}
$$

$$b_1 = \frac{1\times 1 - 1\times \cancel{1}1}{1} = 0 \Rightarrow \varepsilon > 0$$

$$b_2 = \frac{1\times 3 - 1\times 0}{1} = 3$$

$$c_1 = \frac{\varepsilon\times 1 - 1\times 3}{\varepsilon} = \frac{\varepsilon - 3}{\varepsilon}$$

$$c_2 = \frac{\varepsilon\times 0 - 1\times 0}{\varepsilon} = 0$$

$$d_1 = \frac{\dfrac{\cancel{\varepsilon - 3}}{\varepsilon}\times 3 - \varepsilon\times 0}{\dfrac{\cancel{\varepsilon - 3}}{\varepsilon}} = 3$$

$\varepsilon > 0$

$c_1 = \dfrac{\varepsilon - 1}{\varepsilon} < 0$ ε : small positive

changed twice $\Rightarrow$ 2 poles $\in$ RHP

Assume $\varepsilon < 0$ (small)

$\Downarrow$

Two sign changes

$\Rightarrow$ 2 poles $\in$ RHP

Case 2: the whole row is zero or zero occurs in the very last row.

This happens, for example, when all the odd (even) powers of the polynomial are missing:

$$D(s) = (s+2j)(s-2j)(s+3j)(s-3j) = s^4 + 13s^2 + 36$$

$$
\begin{array}{c|ccc}
s^4 & 1 & 13 & 36 \\
s^3 & 0 & 0 & 0
\end{array}
$$

Solution:

1. Construct an auxiliary polynomial based on the row before the all-zero row
2. Replace the all-zero row by the DERIVATIVE of the auxiliary polynomial.

Why? As motivated by the previous example, a zero row implies that a polynomial D(s) has only even or odd power. It turns out in this case, D(s) and D(s)+D'(s) have exactly the same numbers of r.h.p. poles (proof beyond scope). As the goal is just to find the # of r.h.p. poles, we can use D'(s) as a surrogate to continue the procedure.

Example 6-11

$$D(s) = s^7 + 3s^6 + 3s^5 + s^4 + s^3 + 3s^2 + 3s + 1$$

s^7	1	3	1	3
s^6	3	1	3	1
s^5	$b_1 = 8/3$	$b_2 = 0$	$b_3 = 8/3$	
row before where zero $\boxed{s^4}$	$c_1 = 1$	$c_2 = 0$	$c_3 = 1$	
s^3	$d_1 = 4$	$d_2 = 0$		
s^2	$e_1 = \varepsilon > 0$	$e_2 = 1$		
s^1	$f_1 = -4/\varepsilon < 0$			
s^0	$g_1 = 1$			

$\Rightarrow s^4 + s^2 + s^0 = s^4 + s^2 + 1$

$$b_1 = \frac{3\times3-1\times1}{3} = 8/3$$

$$b_2 = \frac{3\times1-1\times3}{3} = 0$$

$$b_3 = \frac{3\times3-1\times1}{3} = 8/3$$

$$c_1 = \frac{8/3\times1-3\times0}{8/3} = 1$$

$$c_2 = \frac{8/3\times3-3\times8/3}{8/3} = 0$$

$$c_3 = \frac{8/3\times1-3\times0}{8/3} = 1$$

Two sign change → two poles on the open right half plane

$$d_1 = \frac{1\times0-8/3\times0}{1} = 0$$

$$d_2 = \frac{1\times8/3-8/3\times1}{1} = 0$$

whole row zero =>

1) last row $s^4 + 0s^2 + 1$

2) $\dfrac{d(s^4+1)}{ds} = 4s^3 + 0$

no $s^1 \Rightarrow 0$

$$e_1 = \frac{4\times0-1\times0}{4} = 0$$

$$e_2 = \frac{4\times1-1\times0}{4} = 1$$

Fist one is zero => $e_1 = \varepsilon > 0$

$$f_1 = \frac{\varepsilon\times0-4\times1}{\varepsilon} = -\frac{4}{\varepsilon}$$

$$\therefore\ g_1 = \frac{-4/\varepsilon\times1-\varepsilon\times0}{-4/\varepsilon} = 1$$

Sample Application of Routh Array: Range of system parameters.

Cannot use numerical method because of the unknown parameter k!

$H(s) = \frac{N(s)}{D(s)}$

Example: $D(s) = s^3 + 3s^2 + 3s + (1+k)$

unknown

$$\begin{array}{lll} s^3 & 1 & 3 \\ s^2 & 3 & 1+k \\ s^1 & b_1 = (8-k)/3 \geq 0 & \\ s^0 & c_1 = 1+k \geq 0 & \end{array}$$

$$b_1 = \frac{3\times 3 - 1\times(1+k)}{3} = \frac{8-k}{3}$$

$$c_1 = \frac{(8-k)/3\times(1+k) - 3\times 0}{(8-K)/3} = 1+k$$

Stable system

$$\left.\begin{array}{l} \frac{8-k}{3} > 0 \Rightarrow 8 > k \\ 1+k > 0 \Rightarrow k > -1 \end{array}\right\} \Rightarrow \boxed{8 > k > -1}$$

to ensure system stable!

$\Downarrow$

No sign change $\Rightarrow$ No poles on the open RHP $\Rightarrow$ marginally stable

what if $\frac{8-k}{3} = 0 \Rightarrow k = 8$

$$\begin{array}{lll} s^3 & 1 & 3 \\ s^2 & 3 & 1+8=9 \\ s & \cancel{0}\to 6 & 0 \\ 1 & 9 & 0 \end{array}$$

$\Rightarrow 3s^2 + 9 \xrightarrow{\frac{d}{ds}} 6s + 0$

$k = 8$ is okay $\Leftarrow$

what if $1+k = 0 \Rightarrow k = -1$

$$\begin{array}{lll} s^3 & 1 & 3 \\ s^2 & 3 & 1+k=0 \\ s & 3 & 0 \\ 1 & \cancel{0}\to 3 & \end{array}$$

$\Rightarrow 3s \xrightarrow{\frac{d}{ds}} 3$

$k = -1$ is okay $\Leftarrow$

The system will be (marginally) stable if

$$-1 \leq k \leq 8$$