Review from last time:

\[ X \perp Y \mid S_1 \cup S_2 \]
\[ \uparrow \quad \uparrow \]
\[ A \quad B \]

\[ X \perp Y \mid S \]
\[ \uparrow \quad \uparrow \]
\[ A \quad B \]

Back to the V-shape example:

Is \( X \perp Z \mid W \)? YES!!

Is \( X \perp Z \mid W \)? NO!!

\( P(W, X, Y, Z) = P(Y \mid X, Z, W) \cdot P(X) \cdot P(Z) \cdot P(W) \)

\( P(W, X, Z) = \sum_Y P(Y \mid X, Z, W) \cdot P(X) \cdot P(Z) \cdot P(W) \)

\( \Rightarrow X \perp Z \perp W \)
General, how can we determine if $X \notin Y \mid X_E$?

Answer: Bayes Ball (Shachter '98) "Rational Passtime"

Goal: Given a DAG, a set evidence nodes $X_E$, a particular node $X_0$
Find all the nodes $\forall X_0 \mid X_E$.

Data Structures:
- Adjacency $(i,j) = 1$ if $X_i \rightarrow X_j$
- Traversed $(i,j) = 1$ if some pt in the edge
- Visited $(i) = 1$ if a ball has visited $X_i$

Ball in a Bag (a dynamic data structure)
- Ball from $= i$
- Ball to $= j$

Step 1:
Create balls for all $j$ such that $\text{adjacency}(0,j) = 1$ or $\text{adjacency}(j,0) = 1$ and put them in the bag.

Set $\text{traversed}(0,1) = 1$, $\text{visited}(1) = 1$
$\text{traversed}(0,2) = 1$, $\text{visited}(2) = 1$

Step 2:
Select the next ball from the bag and follow the rules below.
Let say ball from $= i$, ball to $= j$. We have 4 cases:
Rule a) $X_j \in X_E$ and $X_i \rightarrow X_j \iff \text{adjacency}(i,j) = 1$

For all $k$, if adjacency $(k, j) = 1$ and traversed $(j,k) = 0$

then 1) add ball to beg with ball from $j$, to $k$

2) traversed $(j,k) = 1$, visited $(k) = 1$

This logic also implies $X_i \rightarrow X_j$ where $X_j$ is a leaf.

b) $X_j \in X_E$ and $X_i \rightarrow X_j$

No Bounce Back because

If adjacency $(j,k) = 1$ and traversed $(j,k) = 0$

then add ball...

It also takes care of the leaf node.

c) $X_j \in X_E$ and $X_i \leftarrow X_j$

Boundary case "$X_j$ is the\' leaf node"
d) $X_j \not\perp X_E$ and $X_{ji} \leftarrow X_j$

Step 3: Terminate when there are no balls in the bag

Step 4: Return all the nodes that have not been visited

Ex.

At the end, $X_4$ is NOT conditional independent with any other node given $X_1$ and $X_6$

Ex. Given $X_{ji}$ in a DAG, what is the smallest set of nodes to condition on so that $X_i$ will be conditionally independent from the rest of the graph given this set?

$X_{wi}$ “parents”

$X_c_i$ “children”

$X_{si}$ “spouses” (nodes that share same children with $X_i$)