Where were we?

Need to show JT = Triangulated graph

For JT, we'll use the “decomposable graph” definition:

1. LT + RT are to any separator S are both decomposable.
2. S separates LT & RT

For Triangulated graph, recall definition: a chord exists for any loop.

Step 1: Decomposability $\Rightarrow$ Triangulation

Prove by contradiction.

Assume there is a decomposable clique graph whose underlying graph is not triangulated $\Rightarrow$

there exists a loop in the graph with no chord.
In this loop, select any two nodes $X$ & $Y$ not adjacent to each other:

Since $X$ & $Y$ are not neighbors, they belong to different cliques.

Thus, in the junction tree,

Pick any separator $S$ along the unique path between $C_1$ & $C_2$.

Since $S$ separates the LT & RT, all paths (in the underlying graph) must intersect $S$.

$\Rightarrow \exists$ node $A \in$ upper path and $B \in$ lower path such that $A, B \in S$

But $S$ is an intersection between 2 cliques, and thus must be a clique itself.

$\Rightarrow \quad \bigcirc A \longrightarrow \bigcirc B$

Contradiction!
Step 2: Triangulation $\Rightarrow$ Decomposability

By induction on $\#$ of nodes $N$.

Base case: single node is trivial.

Assumption: all $N$-node triangulated graphs are decomposable.

Consider any $(N+1)$-node graph $G$.

If $G$ is complete, it must be decomposable.

Otherwise, pick $X, Y \in G$ that are not adjacent.

Enumerate all paths between $X$ and $Y$:

Define $S$ as the smallest set of nodes that intersects all paths.

Example: $S = \{U_2, U_6\}$ or $\{U_2, U_7\}$, $S' = \{U_3, U_5, U_7\}$ not smallest.

* For any $U \in S$, there exist a path from $X$ to $Y$ containing $U$ such that $U$ is the only node on the path in $S$.

Why?

Now, remove $S$ from $G_i$ and define

$A = \text{all nodes in } G_i \setminus S \text{ reachable from } X \text{ by paths in } G_i \setminus S$

$B = G_i \setminus S \setminus A = \text{all nodes in } G_i \setminus S \text{ reachable from } Y \text{ by paths in } G_i$

Neither $A$ nor $B$ are empty as $X \in A$, $Y \in B$

Goal: Show $A \cup S = L_I$, $B \cup S = R_I$ and $S = \text{separator}$.
1. $\text{A} \cup \text{BUS}$ clearly covers the entire graph
2. $(\text{AUS}) \cap (\text{BUS}) = S$
3. Are AUS and BUS triangulated? Yes!
   - All connected subgraphs of a triangulated graph are triangulated.
   - AUS and BUS are connected by definition of A + S.
   - They are triangulated $\Rightarrow$ they are decomposable by assumption as they have fewer than $N+1$ nodes.
4. Are AUS and BUS separated by S? Yes!
   If not, $\exists$ a path between A & A and B & B such that
   **Contradiction!!**
5. Are we done?
   No, for S to be a separator, it must be a clique.

Prove by contradiction. Assume there are $S_1$, $S_2$ $\in S$ such that $S_1$ & $S_2$ are not adj.
We have this situation:

If there are shortcuts with $A \cup B$, the loop that involved $S_1$ & $S_2$ is still there. Also no shortcut exists between $A \cup B$ as S separates $A \cup B$. Incorporating all the short cuts in $A \cup B$ will result in a loop with at least 4 nodes $\Rightarrow$ a chord must exist $\Rightarrow$ as there are no more chords in $A$ or $B$, $S_1$ and $S_2$ must be adjacent to each other !!!