Preliminaries

Sample Space \( \Omega \): set of possible outcomes

Sample: \( w \in \Omega \)

Event: subset of \( \Omega \)

Probability measure: \( P: \Omega \to [0,1] \)

1. \( P(A) \geq 0 \)
2. \( P(\Omega) = 1 \)
3. \( A_1, A_2, \ldots \) partition of \( \Omega \) \[ \bigcup_{i=1}^{\infty} A_i = \Omega, A_i \cap A_j = \emptyset \text{ if } i \neq j \]

\[ P\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i) \]

Random Variables \( X: \Omega \to \mathbb{R} (\mathbb{R}^n) \)

\( X = x \iff \text{Event } X^{-1}(x) = \{ w_1, w_2, w_3 \} \)

\[ P(X = x) = P(\{ w_1, w_2, w_3 \}) \]

\( \Omega \)

\( \mathbb{R} \)

Independence: \( X, Y \) are independent \( \iff P(X,Y) = P(X) \cdot P(Y) \)

Conditional Prob: \( P(X | Y) = \frac{P(X,Y)}{P(Y)} \) \( [P(\cdot | Y=y) \text{ is a probability fn}] \)

MWF 3-3:50pm \( \rightarrow \) *MW 3:30-4:45pm

Pass!
Conditional Probabilities

\[ P(X \mid Y) = \frac{P(X, Y)}{P(Y)} \]

- \( P(\cdot \mid Y=y) \) is a probability mass/density function
- Bayes Rule:
  \[ P(X \mid Y) = \frac{P(X, Y)}{P(Y)} = \frac{P(Y \mid X) P(X)}{P(Y)} \]

\[ P(X \mid Y=y) \propto P(Y \mid X) P(X) \quad \text{as} \quad Y \text{ is fixed.} \]

- Law of total probability:
  \[ P(X) = \sum_{y} P(X \mid Y=y) P(Y=y) \]

Why? \( P(X, Y) \quad \overset{?}{\longrightarrow} \quad P(X) = \sum_{y} P(X \mid Y=y) P(Y=y) = \sum_{y} P(X \mid Y=y) P(Y=y) \)

- Conditional Independence

\[ X \perp Y \mid Z : \quad P(X, Y \mid Z) = P(X \mid Z) \cdot P(Y \mid Z) \]

\( \because \)
- \( X = \text{I slipped} \)
- \( Y = \text{raining} \)
- \( Z = \text{the ground is wet} \)

\[ X \perp Y \mid Z \quad \text{if and only if} \quad P(X, Y, Z) = P(X \mid Z) \]

\[ P(X \mid Y, Z) = \frac{P(X, Y, Z)}{P(Y, Z)} = \frac{P(X \mid Y) P(Y) P(Z)}{P(Y) P(Z)} = P(X \mid Z) \quad \# \]
All the relationships can be extended to many R.V.'s:

\[
P(X_1 | X_2, X_3, X_4) = \frac{P(X_1, X_2 | X_3, X_4)}{P(X_2 | X_3, X_4)}
\]

**Expectation:** \( E(X) = \int x f_X(x) \, dx \) (continuous)

\[
= \sum_{\text{all } x} x \, P(X = x)
\]

**Conditional Expectation:** \( E(X | Y) = \sum_{\text{all } x} x \, P(X = x | Y) \) is a function of \( Y \).