Please answer the following questions.

1. (10 points) After your yearly checkup, the doctor has some bad news and good news for you. The bad news is that you tested positive for a serious disease, and that the test is 99% accurate (i.e. the probability of testing positive given that you have the disease is 0.99, as is the probability of testing negative given that you don’t have the disease.) The good news is that this is a rare disease, striking only one in 10,000 people.
   
   (a) Why is it good news that the disease is rare?
   
   (b) What are the chances that you actually have the disease?

2. (20 points) Assume you are a contestant of a game show in which you are presented with three closed doors A, B, and C. Behind one of the doors is a car which will be yours if you choose the right door. After you have randomly (as you have no prior information) selected a door (say door A), the game host opens door B which has nothing inside, while keeping door A and C closed. The host then asks whether you want to change your selection from A to C. Should you change? Explain your answer. [Hint: Consider the sample space of 3-tuples \( \omega = (\omega_1, \omega_2, \omega_3) \) which represents the door you choose, the door with a car behind, and the door the host opens respectively. Calculate and compare \( P(\{\omega : \omega_2 = C\}|\{\omega : \omega_1 = A, \omega_3 = B\}) \) and \( P(\{\omega : \omega_2 = A\}|\{\omega : \omega_1 = A, \omega_3 = B\}) \).]

3. (20 points) Let \((\Omega_1, \mathcal{F}_1, P_1)\) be Borel measure on \((0,1]\). Consider a second probability triple, \((\Omega_2, \mathcal{F}_2, P_2)\), defined as follows: \(\Omega_2 = \{1,2\}\), \(\mathcal{F}_2\) consists of all subsets of \(\Omega_2\), and \(P_2\) is defined by \(P_2\{\{1\}\} = \frac{1}{4}\), \(P_2\{\{2\}\} = \frac{3}{4}\), and additivity. Let \((\Omega, \mathcal{F}, P)\) be the product measure of \((\Omega_1, \mathcal{F}_1, P_1)\) and \((\Omega_2, \mathcal{F}_2, P_2)\).
   
   (a) Express each of \(\Omega\), \(\mathcal{F}\), and \(P\) as explicitly as possible.
   
   (b) Find a set \(A \in \mathcal{F}\) such that \(P(A) = \frac{3}{4}\).

4. (20 points) Give an example of events \(A\), \(B\), and \(C\), each of probability strictly between 0 and 1, such that
   
   (a) \(P(A \cap B) = P(A)P(B), P(A \cap C) = P(A)P(C), \) and \(P(B \cap C) = P(B)P(C)\); but it is not the case that \(P(A \cap B \cap C) = P(A)P(B)P(C)\). [Hint: You can let \(\Omega\) be a set of four equally likely points.]
   
   (b) \(P(A \cap B) = P(A)P(B), P(A \cap C) = P(A)P(C), \) and \(P(A \cap B \cap C) = P(A)P(B)P(C)\); but it is not the case that \(P(B \cap C) = P(B)P(C)\). [Hint: You can let \(\Omega\) be a set of eight equally likely points.]

5. (30 points) Problem 1.31 in Hajek. This is the problem that goes through step by step on how to construct a subset within the unit interval that is not measurable. This is a difficult problem so you may want to discuss with a fellow classmate but everyone should turn in his/her own work.