EE640 Homework #3 (Due February 19, 2013)

This homework set is not straightforward. You are encouraged to work in groups but you must turn in your own work. Also, please start early. Good luck.

1. (10 points) Let $A_1, A_2, \ldots$ be a sequence of events. During lecture, we introduce

$$B_n = \bigcup_{m=n}^{\infty} A_m, \quad C_n = \bigcap_{m=n}^{\infty} A_m$$

Clearly $C_n \subseteq A_n \subseteq B_n$. The sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing with limits

$$\lim_{n \to \infty} B_n = B = \bigcap_{n} B_n = \bigcap_{m \geq n} A_m,$$

$$\lim_{n \to \infty} C_n = C = \bigcup_{n} C_n = \bigcup_{m \geq n} A_m;$$

$B$ and $C$ are called $\{A_n, \text{infinitely often}\}$ and $\{A_n, \text{almost aways}\}$ respectively. Show that

(a) If $B = C$, we say that $A_n$ converges to the limit $B$ (or $C$). Show that this limit is a member of the $\sigma$-algebra.

(b) Show that $\lim_{n \to \infty} P(A_n) = P(B)$.

2. (10 points) $\{A_n, \text{infinitely often}\}$ and $\{A_n, \text{almost aways}\}$ are examples of tail events which are defined to be those events that involve infinitely many $A_n$’s. Very often, tail events have probability either 0 or 1, depending on probabilities of individual $A_n$. If the probabilities of individual $A_n$ can be controlled as a design parameter, adjusting this parameter may result in an abrupt change of the probability of the tail event. Such phenomenon is called phase transition and occurs in many natural settings and engineering designs. One such example is connectivity in wireless ad hoc networks. Find and read the paper “Phase Transition Phenomena in Wireless Ad Hoc Networks” by Krishnamachari, Wicker, and Bejar. Write a short summary of the paper.

3. (10 points) Show that a distribution function has at most countably many discontinuities. Recall that at a discontinuity $x_o$, $F_X(x_o) \neq F_X(x_o)$ or $P(X = x_o) > 0$.

4. (10 points) Let $Z = \min\{X_1, \ldots, X_n\}$ where $X_1, \ldots, X_n$ are independent identically distributed random variables, each with distribution function $F_X$.

(a) Show that $F_Z(z) = 1 - [1 - F_X(z)]^n$ for all $z \in \mathbb{R}$.

(b) If $F_X$ has continuous pdf $p_X$, find the pdf for $Z$ in terms of $p_X$.

5. (20 points) You are given a relatively slow Ethernet switch such that it can accepts at most one packet in one millisecond. Let $V_i$ be a Bernoulli random variable with probability $p$ such that $V_i = 1$ represents the event of a packet arriving at the $i^{th}$ millisecond. Consider the random variable $S_n$ which is the number of packets arrived in an $n$-millisecond period; i.e. $S_n = V_1 + \ldots + V_n$ where $V = (V_1, \ldots, V_n)$ are $n$ independent Bernoulli arrivals. Show that the conditional distribution $P(V|S_n = k)$ is uniform on its (reduced) sample space $H_{n,k} \triangleq \{0, 1\}^n \cap \{S_n = k\}$ or more specifically:

$$H_{n,k} = \{v : v = (v_1, \ldots, v_n), \text{where each } v_i = 0 \text{ or } 1 \text{ and } v_1 + \ldots + v_n = k\}$$

This fact should surprise you: this means that given the total number of packet arrivals, all consistent histories of packet arrivals are equally likely regardless of the arrival probability $p$!

6. (20 points) Planet $X$ is a ball with center at $O$. Three spaceships $A$, $B$ and $C$ land at random on its surface, their positions being independent and each uniformly distributed on the surface. Spaceships $A$ and $B$ can communicate directly by radio if $\angle AOB < 90^\circ$ and so do $A$ and $C$ as well as $B$ and $C$. Show that the probability that they can keep in touch (with, for example, $A$ communicating with $B$ via $C$ if necessary) is $\frac{\pi}{4\pi}$. 

Work. Also, please start early. Good luck.
7. (20 points) Using Riemann and Lebesgue approximations usually result in very different results. In this problem, you are asked to write two simple Matlab routines to calculate and compare these two approximations. The definitions of the two routines are as follows:

\[
\begin{align*}
\text{area} &= \text{riemann}(x\_\text{values}, y\_\text{values}, \text{numberOfdivisions}) \\
\text{area} &= \text{lebesgue}(x\_\text{values}, y\_\text{values}, \text{numberOfdivisions})
\end{align*}
\]

\text{x\_values} and \text{y\_values} denote the piecewise linear function \( f(x) \) of which you need to approximate its integral. You can assume that \( f(x) \) is non-negative. \text{numberOfdivisions} defines the number of regular divisions in \( x \)-axis (for Riemann) or \( y \)-axis (for Lebesgue) used in the approximation. You are asked to compute the approximations using four different numbers of divisions (5, 10, 20, 100) for a sine signal and a chirp signal defined below. Compare the accuracy of the two approximations against the true results (either by hand or some numerical package).

(a) Sine signal: \( y = 0.5 + \sin(20\pi x) + 0.5 \) for \( 0 \leq x \leq 1 \) or in Matlab:
\[
\begin{align*}
x\_\text{values} &= 0:0.001:1; \quad y\_\text{values} = 0.5*\sin(20*\pi*x\_\text{values})+0.5.
\end{align*}
\]

(b) Chirp signal: \( y = 0.5 + \sin[20\pi(0.5 + 2x)x] + 0.5 \) for \( 0 \leq x \leq 1 \) or in Matlab:
\[
\begin{align*}
x\_\text{values} &= 0:0.001:1; \quad y\_\text{values} = 0.5*\sin(20*\pi*(0.5+2*x\_\text{values})*x\_\text{values})+0.5.
\end{align*}
\]