EE640 Homework #5 (Due March 26, 2013)

You are encouraged to work in groups but you must turn in your own work. Also, please start early. Good luck.

1. (20 points) Basic MMSE

Define the probability mass function $p : \Omega \to \mathbb{R}$ by

$$p(\omega) = 3, 2, 1, 1, 2, 1$$

for $\omega \in \Omega = \{1, \ldots, 6\}$

Define random variable $X : \Omega \to \mathbb{R}$ by $X(\omega) = (\omega - 4)^2$. We are interested in measuring $X$ but a coarse equipment can return the following measurement variable $Y : \Omega \to \mathbb{R}$:

$$Y(\omega) = 0, 2.5, 2.5, 8, 8, 8$$

for $\omega = 1, \ldots, 6$

(a) Interpret $E(X|Y)$ as a function of $Y$ values or $f(Y)$. Verify that $EX = E(E(X|Y))$.

(b) Interpret $E(X|Y)$ as a function on $\Omega$, so $E(X|Y)$ has value $f(Y(\omega))$ at $\omega$. Plot $E(X|Y)$ and $X$ over the $\omega$-axis. Use the plots to explain why $E(X|Y)$ minimizes the square error $E((X - f(Y))^2)$ among any Borel measurable function $f$.

(c) Use the projection theorem to prove that $E[E(X|Y,Z)|Y] = E(X|Y)$.

(d) Use the definition of conditional expectation to prove that $E[E(Xg(Y)|Y)] = g(Y)E(X|Y)$.

2. (20 points) More on conditional expectation

(a) Prove the “tower property” of conditional expectation: i.e $E[E(X|Y)] = EX$.

(b) $X_1, X_2, \ldots$ are a sequence of independent identically-distributed random variables with mean $E(X)$ and variance $Var(X)$. You are asked to randomly select a positive integer $N$ and compute the expectation and variance of $Y_N = \sum_{i=1}^{N} X_i$. Hint: Use part a.

(c) Will your answer to part b change if $N$ is not independent of $X_1, X_2, \ldots$?

3. (20 points) LLSE

Let $U = (X_1 X_2 Y)^T$ where

$$X_2 = X_1 + W$$

$$Y = X_1 + W + Z$$

and $V = (X_1 W Z)^T$ has $E(V) = 0$ and $E(VV^T) = I$.

(a) Exhibit $E(UU^T)$.

(b) Find the LLSE of $Y$ given $X_1$.

(c) Find the LLSE of $Y$ given $X_1$ and $X_2$.

(d) The last two parts show that the LLSE based on $X_1$ may utilize $X_1$ but when $X_2$ is also available then $X_1$ is no longer needed. You might have predicted this at the outset. Why?

4. (20 points) LLSE vs. MMSE

(a) Let $X$ and $Y$ be IID and exponentially distributed mean 1. Calculate $E[X|\min(X,Y)]$ and the LLSE $E(X|\min(X,Y))$.

(b) Let $W$ be a standard Gaussian $N(0,1)$ random variables. For each of the following cases, find the MMSE $E(V|U)$ and the LMSE $E(V|U)$:

i. $V = W^3$ and $U = W$

ii. $V = W^2$ and $U = W^3$
5. (20 points) **Multivariate Gaussian**

Let $\mathbf{Y} = (Y_1 \ Y_2 \ \ldots \ Y_m)^T$ be a Gaussian vector with mean $\mu_Y$ and covariance matrix $C_Y$, and $\mathbf{X} = (X_1 \ X_2 \ \ldots \ X_k)^T$ be another Gaussian vector with mean $\mu_X$ and covariance $C_X$. Let $\mathbf{Z} = (\mathbf{X}^T \mathbf{Y}^T)^T$. Assume that $\mathbf{Z}$ is jointly Gaussian, we know that the mean and covariance of $\mathbf{Z}$ can be expressed as follows:

$$E(\mathbf{Z}) = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

$$C_Z = \begin{bmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{bmatrix}$$

where $C_{XY} = \text{Cov}(\mathbf{X}, \mathbf{Y}) = C_Y^T X$. 

(a) Let $\hat{\mathbf{X}}(\mathbf{Y}) = \mu_X + C_{XY} C_Y^{-1}(\mathbf{Y} - \mu_Y)$. Show that $\mathbf{X} - \hat{\mathbf{X}}(\mathbf{Y})$ and $\hat{\mathbf{X}}(\mathbf{Y})$ are independent of each other.

(b) Show that the conditional distribution of $\mathbf{X}$ given $\mathbf{Y}$ is jointly Gaussian with mean $\mu_{X|Y} = \hat{\mathbf{X}}(\mathbf{Y})$ and covariance $C_{X|Y} = C_X - C_{XY} C_Y^{-1} C_{YX}$. 
