1. Stochastic Processes
   - How to do differentiation & integration on $X(t,\omega)$?

2. Ergodicity (7.4)
   - Is time average equal to statistical average?

3. Stochastic Linear System
   - Interest in computing $R_{xy}$ and $R_{yy}$.

4. Noise Removal and Channel Equalization
   - Very easy

5. (9.1-9.5)

We need a few things:

- Focus on Wide-sense Stationary Process $i.e. R(x(t),x(t-s)) = E(x(t)x(t-s)) = H(x(t-s))$.
We have seen this in Brownian motion. Use the definition of reducible to discrete for any $t$. In fact, for any sense $t \rightarrow t'$ no longer discrete...

\[ \lim_{t \to t'} (X_t) = X_{t'} \]

Continuity of Extensions

Focus here: mean square due to the mean square prediction problem.

Here, mean square $\Rightarrow$ probability distribution $\Rightarrow$ almost surely $\Rightarrow$ different modes.

In Chapter 2, we learn $X_{n|t'}(t) \to X_{n|t'}(t)$.
Since we are considering the case $S=2$, it is sufficient for $R(x,t)$ to be continuous.

$\mathbb{E}(x^2 - x^2_t) = \mathbb{E}x^2_t - 2\mathbb{E}x^*_t + \mathbb{E}x^2 = Rx(s, s) - 2Rx(s, t) + Rx(t, t)$

To show $X(t)$ to be continuous at $t=0$, we need $\mathbb{E}(x^2 - x^2_t) \to 0$ as $s \to t$.

Idea of the proof:

$LHS: R_x(t, t)$ is continuous for all $t$.

$R_x(t, t)$ is continuous for all $t$.

In fact, if we focus on $R_x(t, t)$, it is sufficient to look at just $R_x(t, t)$. 

DEFINITION
Corollary 2.1.5 \( (p.55) \): \( X_n \xrightarrow{m.s.} X \iff E(X_n) \xrightarrow{m.s.} E(X) \) as \( n \to \infty \).

\[
\begin{align*}
5.5 - t, & \quad s_2 \to \frac{x - X}{s_2} \\
5.5 - t, & \quad s_3 \to \frac{x - X}{s_3} \\
5.5 - t, & \quad s_4 \to \frac{x - X}{s_4} \\
\end{align*}
\]

Idea of Proof: For each \( t \), pick \( S_r \).

Characteristic \( \frac{d}{dt} \)

\[ \Rightarrow \]

Theorem: \( E(X(t)) = \frac{d}{dt} E(X(t)) \) if Mean of Derivative = Derivative of Mean

\[ \frac{d}{dt} \] is a random process.

\[ X(t) \] is differentiable at \( t \_0 \) in m.s. sense \( \iff X(t) \xrightarrow{m.s.} X(t) \) when \( s \to 0 \). Differentiability. \[ \frac{d}{dt} \]
In particular, if \( \mu_t \) is WSS, \( \mu(t) = \frac{\partial}{\partial t} \mu(t) = 0 \) is constant.

\[ \begin{align*}
&\text{A.S.} \quad \begin{cases}
\frac{\partial}{\partial t} E(x_t) = E(x_t) \\
\mu(x_t) = \frac{\partial}{\partial t} \mu(x_t) = 0
\end{cases}
\end{align*} \]

This implies very nice because we can now talk about the statistics of \( x_t \).

Thus, we have:

\[ E(x_t) \]
\[ R_{xt} = \frac{10\cdot R_x}{a} \]

where \( s = t + 3 \).

Thus, \( s \) is a constant with \( s = 2t + 3 \).

The cross-correlation between \( x_t \) and \( x_t \) is:

\[ R_{xt}(t, t+3) \]
\[ x^T \text{ is also WSS} \]

In particular, for WSS \( x^T \) \( R_X(s, t) = \frac{\sigma^2}{\theta^2} R_X(t) \) with \( t = s - t \)

\[ \frac{\sigma^2}{\theta^2} R_X(s, t) = \frac{\sigma^2}{\theta^2} R_X(s - t) \]

or \( \sigma^2 \) \( R_X(s, t) = \mu (x, x') = \mu (s, t) \)

\[ (1) \quad R_{XX}(t) = \frac{\sigma^2}{\theta^2} R_X(t) \]

with \( t = s - t \)

But for WSS \( x^T \)

\[ \frac{\sigma^2}{\theta^2} R_X(s, t) = \frac{\sigma^2}{\theta^2} R_X(s - t) \]

by similar argument.
For simplicity, we will use Riemann Sum. 

\[ \int_a^b f(x) dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(t_k) \Delta x \]

Area = \( (t_{k+1} - t_k) \cdot x_k \cdot h(t_k) \)
Also, similar to $\frac{d}{dx}$, integration and expectation are also commutative.

$$
\mathbb{E} \left[ \frac{1}{t} \sin(wt + \theta) \right] = \frac{1}{t} \int_{-\infty}^{\infty} \sin(wt + \theta) dt
$$

as $\theta$ is constant w.r.t $t$.

$\mathbb{E}$ is defined as

$$
\mathbb{E} = \int_{-\infty}^{\infty} x dt = \int_{-\infty}^{\infty} \cos(wt + \theta) dt
$$

ex. let $X_t = \cos(wt + \theta)$ where $\theta \sim \mathcal{U}(0, 2\pi)$

In most cases, stochastic integration behaves exactly like deterministic ones.
\[ \int_0^T R^x(t) \, dt = \int_0^T R^{x(y)}(t) \, dt \]

\[ \int_0^T R^{x(y)}(-t) \, dt = \int_0^T R^{x(y)}(t) \, dt \]

\[ \int_0^T R^{x(y)}(t) \, dt = \int_0^T R^{x(y)}(t + T) \, dt \]

If \( x \) is WSS, \( x[Y] = x[Y(T)] \) and \( x[Y] = x[Y(t)] \)

\[ \int_0^T E(x) \, dt = \int_0^T \frac{1}{2} \left( \int_0^T x(t) \, dt \right) E \, dt \]

Let \( y \in \mathcal{A} \)
(Summarized: White noise is fundamentally in modern system noise.)

\[ \mathcal{X}_t \text{ is unobservable} \]

But this is not a second order process as \( \mathbb{E}(X^2_t) = \infty \). (Infinite power)

\[
\int_{0}^{1} \mathbb{E}(X_t^2) dt = \mathbb{E}(\mathbb{E}(X_t^2 | h(t)))
\]

\[ \mathbb{E}(X^2_t) = 0 \text{ and } \mathbb{E}(X_t) = 0 \text{ for all } t > 0. \]

In continuous-time, can we define something similar?

Recall the white noise process in discrete-time.

\[
\mathbb{E}(X_n) = 0, \quad \mathbb{E}(X_n^2) = 6 \pi^2 (n)
\]

White noise process
What is \( \gamma \)?

\[
\gamma = b_{\min} (5, t)
\]

\[
= \int_0^1 \{ 1 + \int_t^1 \text{v} \, dt \} \, dx
\]

\[
= \int_0^1 \{ 1 + \text{v} \} \, dx
\]

\[
= \int_0^1 \text{v} \, dx
\]

\[
\Rightarrow \quad \gamma (s, t) = \int_0^1 \text{v} \, dx
\]

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\]

\[
\Rightarrow \quad \gamma (s, t) = \int_0^1 \text{v} \, dx
\]

Let's assume for the moment \( \gamma \) could be realised ... (say in Twilight Zone)
Thus, in continuous-time modeling, the white noise process is typically absorbed into the integral. This integral is called the Itô integral.

\[ Y_t = \int_0^t f(t) \, dW_t \]

The output of a system with white noise input is not defined.
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu \quad \text{(continuous-time)} \]

\[ \frac{1}{t} \int_0^t x(t) \, dt \]

And you might take

\[ \frac{1}{\sqrt{t}} \int_0^t x(t) \, dt \]

Run the experiment and collect a sample path.

\[ \text{WSS } x(t) \text{ with unknown } \mu \text{ that you want to estimate} \]
\[
R^{\text{WM}(m,n)} = \mathbb{E}[\text{WM}(m,n)] = \mathbb{E}[\text{WM}(m)] + \mathbb{E}[\text{WM}(n)] + \mathbb{E}[\text{WM}(m,n)]
\]

\[
= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{2}{4} = \frac{1}{2}
\]

\[
\text{WN is WSS : } \mathbb{E}(\text{WN}) = \mathbb{E}[\text{WN}^2] + \mathbb{E} [\text{WN}] = \mathbb{E}[1 - 5](\text{WN}) + \mathbb{E}(\text{WN})
\]

\[
\text{True random process observed}
\]

\[
\text{Select one at random in the beginning : } S = 0 \text{ or } S = 1
\]

\[
\frac{P(\text{WN}=1)}{2} = \frac{1}{2}
\]

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\frac{P(\text{WN}=1)}{2} = \frac{1}{2}
\]

Not all WSS exist in mean ergodic.
Thus
\[
\lim_{N \to \infty} W_N = \frac{1}{\epsilon} \sum_{i=1}^{N-1} W_i = \frac{1}{\epsilon} \sum_{i=1}^{N-1} \frac{1}{\epsilon} \sum_{j=1}^{i} \sigma_{ij}
\]
and
\[
\frac{1}{\epsilon} \sum_{i=1}^{N} \sigma_{ii} = \frac{1}{\epsilon} \sum_{i=1}^{N} \frac{1}{\epsilon} \sum_{j=1}^{i} \sigma_{ij} \leq \frac{1}{\epsilon} \sum_{i=1}^{N} \frac{1}{\epsilon} \sum_{i=1}^{i} \sigma_{ij} = \frac{1}{\epsilon} \sum_{i=1}^{N} \frac{1}{\epsilon} \sum_{i=1}^{i} \sigma_{ij}
\]
As \( T \to \infty \),
\[
\frac{1}{N} \sum_{i=1}^{N} \left[ \left( \sum_{j=1}^{i} \sigma_{ij} \right) - \frac{1}{N} \right] (\sigma - 1)
\]
\[
\frac{1}{N} \sum_{i=1}^{N} \left[ \left( \sum_{j=1}^{i} \sigma_{ij} \right) - \frac{1}{N} \right] (\sigma - 1)
\]
Time average:
\[
\frac{1}{6} + \left[ \frac{16}{3} (m-w) \right] = Rx(m,m) = Rx(m,w)
\]
\[
\begin{align*}
&u = m \\
&u \neq m
\end{align*}
\]
\[
\frac{1}{6} + \left[ \frac{16}{3} (m-w) \right] = E[\sigma_{m,m}] + E[\sigma_{m,w}] = E[\sigma_{m,w}]
\]
(See proof of Prop 7.4.1.)

\[ \lim_{t \to \infty} \int_{1}^{t} \frac{1}{2} \left( 1 - \frac{1}{t} \right) \mu(x) \, dx = 0 \]

\[ \iff \lim_{t \to \infty} \int_{1}^{t} x^2 \, dx \to \frac{1}{2} \]

To obtain an enough and sufficient condition for mean-ergodic, we need

\[ \lim_{n \to \infty} \frac{1}{2} \sum_{k=1}^{n} x_k = 0 \]

(Here example \( \lim_{n \to \infty} \frac{1}{2} \sum_{k=1}^{n} x_k = 0 \))

\[ \lim_{t \to \infty} x(t) = 0 \]

\( \iff \lim_{t \to \infty} R_x(t) = 0 \) (Because unreached for infinitely far away)

\[ \text{Thus, it is important that the impact of each sample } x(t) \text{ does not last forever.} \]

\[ \text{Set } x(t) \text{ based on a single sample path } \]

\[ \text{Why not mean ergodic? The initial decision is never dismissed.} \]