Convergence of sequence & series

\[ X = \begin{cases} 1 & \text{head w/ unknown prob } p \\ 0 & \text{tail} \end{cases} \]

To estimate \( p \), independent trials \( X_1, X_2, X_3, \ldots \)

\[ \frac{1}{N} \sum_{i=1}^{N} X_i \rightarrow p \quad \text{as } N \to \infty \]

- What type of convergence?
- How to characterize \( \frac{1}{N} \sum_{i=1}^{N} X_i - p \)?

Limit cannot tell the whole story

\[ X_n = \begin{cases} 1 - \frac{1}{n} & \text{if } n \text{ is not a power of 2} \\ 0 & \text{otherwise} \end{cases} \]

\[ \lim_{n \to \infty} X_n \text{ does not converge} \]

Try something else:

\[ \limsup_{n \to \infty} X_n = \lim_{n \to \infty} \sup \{|X_k: k \geq n|\} \]

What is sup? Almost the same as max.

(supremum)
$$S = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\} \quad \max(S) \text{ does not exist.}$$

$$\sup(S) \equiv \min\{c : c \geq a, \forall a \in S\}$$

$$\Rightarrow \sup(S) = 1$$

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup\{x_k, k \geq n\} = 1$$

$$\Downarrow$$

$$y_n = 1, 1, \frac{1}{2}, \ldots$$

$$\uparrow \uparrow \uparrow$$

$$y_1, y_2, y_3$$

\text{limsup always exists (unlike limit)}$$

\text{ex/} \quad x_n = \begin{cases} 
1 - \frac{1}{n} & n \text{ is not power of 2} \\
2 & \text{otherwise}
\end{cases}$$

\text{lim sup } x_n = 2
Use $\liminf_{n \to \infty} x_n \equiv \lim_{n \to \infty} \inf \{ x_k : k \geq n \} = \lim_{n \to \infty} \max \{ c : c \leq x_k, k \geq n \}$

\[ y_n \]

\[ y_1 = \max \{ c : c \leq x_k, k \geq 1 \} = 0 \]

\[ y_2 = \max \{ c : c \leq x_k, k \geq 2 \} = \frac{2}{3} \]

\[ y_3 = \ldots = \frac{2}{3} \]

\[ y_4 = \ldots = 1 - \frac{1}{5} = \frac{4}{5} \]

\[ \vdots \]

\[ \liminf_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 1 \]

Just like $\limsup$, $\liminf$ always exists and is useful to study the asymptotic behavior of divergent sequences.

Continuity of $f$ at $x_0$ can be defined based on limit.

Step 1°/ \[ \lim_{x \to x_0} f(x) = c \iff \text{For all possible sequences } x_n \text{ with } x_n \to x_0, \text{ we have } \lim_{n \to \infty} x_n = x_0, \text{ we have } \lim_{n \to \infty} f(x_n) = c \]
Step 2° $f$ is continuous $\iff \lim_{x \to x_0} f(x) = f(x_0)$

Derivative,

$f$ is differentiable at $x_0$ $\iff \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.

Chapter 1 Yes, finally!

Axiomatic of Probability

A Probability Space $= (\Omega, \mathcal{F}, P)$ "triplet"

$\Omega =$ sample space which consists of all possible outcomes of the phenomenon you are studying

individual outcomes $w_1, w_2, \ldots \in \Omega$

$\mathcal{F} =$ collection of subsets of $\Omega$ "$\sigma$-field" or "$\sigma$-algebra"

$A \in \mathcal{F} \Rightarrow A \subseteq \Omega$ is called an event

We are going to measure probability of an event.
Defining properties of a σ-field $\mathcal{F}$

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c = \Omega \setminus A \in \mathcal{F}$
3. If $A_1, A_2, A_3, \ldots \in \mathcal{F}$ and countable, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

$\mathcal{P} =$ a probability measure.

$\mathcal{P} : \mathcal{F} \rightarrow [0, 1]$ must obey the following properties

\[
\begin{align*}
\text{(1)} & \quad \mathcal{P}(\Omega) = 1 \\
\text{(2)} & \quad A \in \mathcal{F} \Rightarrow \mathcal{P}(A) \geq 0 \\
\text{(5)} & \quad \text{countably additive.} \\
& \quad \text{If } A_1, A_2, A_3, \ldots \in \mathcal{F} \text{ AND they are disjoint } (A_i \cap A_j = \emptyset \text{ if } i \neq j) \\
& \quad \Rightarrow \mathcal{P} \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mathcal{P}(A_n)
\end{align*}
\]

natural

definition

of a

physical

measure

like length, area, ...

\[ \text{Area } \left[ \begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
A_5 \\
A_6 \\
A_7 \\
\end{array} \right] = \text{Area } \left[ \begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
A_5 \\
A_6 \\
A_7 \\
\end{array} \right] + \text{Area } \left[ \begin{array}{c}
A_2 \\
A_3 \\
A_4 \\
A_6 \\
A_7 \\
\end{array} \right] + \ldots \]
Tossing a fair coin once

\[ \Omega = \{ H, T \} \]

\[ F = \{ \Omega, \emptyset, \{ H \}, \{ T \} \} \]  --- all possible subset of \( \Omega \)

\[ \text{ex finite set} \]

\[ A = \{ x_1, x_2, \ldots, x_N \} \]

Axiom: \( P(\Omega) = 1 \)

Definition: \( P(\{H\}) = \frac{1}{2} \)

\[ P(\{T\}) = \frac{1}{2} \]

\[ P(\emptyset) = 0 \] why?

\[ P(\Omega) = P(\Omega \cup \emptyset) \]

\[ = P(\Omega) + P(\emptyset) \]

Since \( \Omega \cup \emptyset = \Omega \), they are disjoint

\[ \Rightarrow P(\emptyset) = 0 \]

Fair Dice: \( S = \{ 1, 2, 3, 4, 5, 6 \} \)

\[ F = 2^S \]

For \( A \in F \) \( \Rightarrow P(A) = \frac{1}{6} \cdot |A| \)
Poisson($\lambda$) - # of letters that you receive today

$\Omega = \{0, 1, 2, 3, \ldots\}$

$p\{w=k\} = \frac{\lambda^k e^{-\lambda}}{k!}$

$q = ?$