HW1 due today (now or my mailbox @ 453E by 5pm)
HW2 out tomorrow

1. \( \Omega \) contains all possible outcomes and should not be affected by the probability of individual outcomes.

2. Given \( \Omega \), there are many possible \( \mathcal{F} \).

\[
U = \text{ or } \quad \bigwedge = \text{ and }
\]

\( \mathcal{F} = \{ \emptyset, \Omega \} \) "trivial" - useless

Useful rule, at the very least, \( \mathcal{F} \) should contain all the individual outcomes in \( \Omega \)

or \( \mathcal{F} \) contains \( \{ w \} \) for every \( w \in \Omega \)

\[ \Rightarrow \text{ Axioms of } \mathcal{F} \text{ will then SIGNIFICANTLY expand } \mathcal{F}. \]

\emph{Example:} \( \Omega = \{ 0, 1, 2, \ldots \} \)

\[
P(\{k\}) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{"Poisson distribution"} \Rightarrow \sum_{k=0}^{\infty} P(\{k\}) = 1
\]

\( \lambda \) - rate

\( \mathcal{F} \) will include all \( \{k\} \) with \( k \in \Omega \)

Since any subset of \( \Omega \) is a countable union of \( \{k\} \)'s

\[ \Rightarrow \mathcal{F} = 2^\Omega \]
How big is $\Omega$?

\[ \Omega = \{ 0, 1, 2, 3, 4, 5, \ldots \} \]

\[ A \cap \Omega = \sqrt{1} \times \times \sqrt{1} \times \times \]

\[ \Rightarrow x_A = 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad \ldots \quad \text{BASE 2} \]

\[ \Leftrightarrow x_A \in [0, 1) \]

This is a bijection! \[ \Rightarrow |\Omega| = |2^\mathbb{N}| = |[0, 1)| \]

How do we assign prob to any $A \subset \Omega$?

Natural\[ P(A) = P\left( \bigcup_{w \in A} \{w\} \right) \]

\[ = \sum_{w \in A} P(\{w\}) \]

If $A$ is finite, $P(A)$ is obviously finite and less than or equal to 1

If $A$ infinite (say $A = \{\text{even numbers}\}$)

\[ P(A) \leq P\left( \bigcup_{k=1}^{\infty} \{k\} \right) = 1. \]
Uniform distribution on unit interval $\Omega = (0,1]$

$$J = \{ (a,b) \text{ with } 0 \leq a \leq b \leq 1 \}$$

$$P(\omega \in (a,b]) = b-a \quad \text{"length"}$$

$\mathcal{F} \supset J \implies \text{expand to include \"closed interval\" } [a,b]$

in my $\mathcal{F}$. 

$$\therefore \quad [a,b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b] \quad \text{st. } a - \frac{1}{n} > 0$$

Since $[a,b] \in \mathcal{F} \implies [a,a] = \{a\} \in \mathcal{F}$ singleton is in!

$$P(\{a\}) = P([a,a]) = a - a = 0$$

$$\sum_{a \in \Omega} P(\{a\}) = \sum_{a \in \Omega} 0 = 0 \neq P(\bigcup_{a \in \Omega} \{a\}) = P(\Omega) = 1$$

$\Rightarrow$ does not violate the axiom because $\Omega$ is uncountable

Think $\frac{1}{\text{cm}}$ but the length of each point is 0

How big is $\mathcal{F}$? Is $\mathcal{F} = 2^{\{0,1\}}$?

NO. $\mathcal{F}$ cannot be $2^{\{0,1\}}$!
Thm: There exists a subset $H \subset (0,1)$ which has no length.

In HW, you will find $H$ such that

$$\sum_{n=1}^{\infty} P(H) = 1 \Rightarrow \infty \cdot P(H) = 1$$

So what is $P(H)$? $P(H)$ cannot be 0 as $\sum_{n=1}^{\infty} 0 = 0$.

$P(H)$ cannot be any tree as $\sum_{n=1}^{\infty} \infty = \infty$.

$\Rightarrow H$ CANNOT BE IN $\mathcal{F}$

How do we define $\mathcal{F}$ for $(0,1)$?

$\mathcal{J} = \{(a,b) : 0 \leq a \leq b \leq 1\}$

"semi-algebra"

$P((a,b)) = b - a$

Countably additive prob.

Categorical Extension

Theorem

"Constructive"

$\mathcal{B}(J) =$ smallest $\sigma$-field that contains $\mathcal{J}$

$= \text{Borel } \sigma$-algebra

$P^* : \mathcal{B}(J) \to [0,1]$ such that

$\forall A \in \mathcal{J} : P^*(A) = P(A)$

for length on intervals. $P^*$ is called Lebesgue measure.
Flip a coin infinite # of times. Biased coin $P(H) = p$

$$\Omega = \{ \text{HHTHH...}, \text{TTTHH...}, \ldots \} \quad P(T) = 1 - p$$

$$0.0010\ldots_2 \quad 0.1110\ldots_2$$

$$= [0, 1) \quad \Omega \text{ is uncountable}$$

All we need to do is to come up with a semi-algebra and a countably additive prob that match the intuitive

$$\square \text{CET} \quad \Rightarrow \quad (\Omega, \mathcal{E}(\Omega), P^*)$$

Semi-algebra:

$$A_H = \{ w : w = H^{\infty} \ldots \} \Rightarrow P(A_H) = p$$

$$A_T = \{ w : w = T^{\infty} \ldots \} \Rightarrow P(A_T) = 1 - p$$

$$A_H \cup A_T = \Omega$$

In fact: $$A_H = [0, 0.5) \quad \text{and} \quad A_T = [0.5, 1)$$

$$A_{HH} = \{ w : w = \text{HH}^{\infty} \ldots \} \Rightarrow P(A_{HH}) = p^2$$

$$A_{HT} = \{ w : w = \text{HT}^{\infty} \ldots \} \Rightarrow P(A_{HT}) = p(1-p)$$

$$A_{TH} = \{ w : w = \text{TH}^{\infty} \ldots \} \Rightarrow P(A_{TH}) = p(1-p)$$

$$A_{TT} = \{ w : w = \text{TT}^{\infty} \ldots \} \Rightarrow P(A_{TT}) = (1-p)^2$$

In fact: $$A_{HH} = [0, 0.25) \quad A_{HT} = [0.25, 0.5) \quad A_{TH} = [0.5, 0.75) \quad A_{TT} = [0.75, 1)$$