HW#2 due next Tuesday
HW#1 solution posted on Bb.

\( \Omega = \{ w : \text{toss a coin infinite \# of times} \} \)

\[ |\Omega| = |\mathbb{R}| \Rightarrow 2^\omega \text{ is way too big for us to define a "sensible" probability measure} \]

Strategy: 1. Define a collection of events that is easy to handle and also has a sensible way to define probability on them

\( \Rightarrow \text{semi-algebra: } J \subset 2^\Omega \text{ obeys 2 properties} \)

1. \( A, B \in J \Rightarrow A \cap B \in J \) "closed under finite intersection"
2. \( A \in J \Rightarrow \exists B_1, B_2, \ldots, B_n \in J \text{ s.t. } A = \bigcup_{k=1}^{n} B_k \) "semi-closed under set difference"

In our example:

\[
\begin{align*}
A_H &= \{ w : w = H*** \ldots \} \quad P(A_H) = P \\
A_T &= \{ w : w = T*** \ldots \} \quad P(A_T) = 1 - P \\
A_{HH} &= \{ w : w = HH*** \ldots \} \quad P(A_{HH}) = p^2 \\
A_{HT} &= \{ w : w = HT*** \ldots \} \quad P(A_{HT}) = p(1-p)
\end{align*}
\]

\( J \leftarrow \begin{cases} 
A_H \\
A_T \\
A_{HH} \\
A_{HT} \\
\vdots \\
\vdots 
\end{cases} \)

\( \Rightarrow J \) is a semi-algebra (skip proof)

\( \Rightarrow P \) obeys countably additive property

2. Done because CET \( \Rightarrow (\Omega, \mathcal{B}(J), P^*) \)

\( P^*: \mathcal{B}(J) \rightarrow [0,1] \)

extension of \( P \) that agrees with \( P \) on everything in \( J \)

smallest \( \sigma \)-field that contains \( J \)
Note that \( A_H = [0.5, 1) \), \( A_T = [0, 0.5) \) by mapping \( H \rightarrow 1 \), \( T \rightarrow 0 \)
\( A_{HH} = [0.75, 1) \), \( A_{HT} = [0.25, 0.75) \)
\( A_{TH} = [0.25, 0.5) \), \( A_{TT} = [0, 0.25) \)

... using \( P \) in this example, 
- "length of \([0.5, 1)\)" \( \subseteq P(A_H) = p \)
- "length of \([0, 0.5)\)" \( \subseteq P(A_T) = 1-p \)

- different from uniform (everyday notion of length)
- but a well-defined measure

\( \Rightarrow \) affect our way to do integration

\[
\text{old way: } \int_0^1 X(w) \, dw \quad \text{new way: } \int \limits_0^1 \underbrace{X(w)}_{\text{uniform}} \, P(dw)
\]

- no need for probability density function at all
- general definition of expectation

MORE LATER ...

\((\Omega, \mathcal{F}, P)\) \( \Rightarrow \) define a probability space on
\((\Omega_1, \mathcal{F}_1, P_1) \times \Omega_2 \) "Product space"

\(\Omega = \{ \text{a single coin toss} \}\)
\(\mathcal{G} = \{ \emptyset, \Omega, \{ H \}, \{ T \} \}\)
Consider $\Omega \times \Omega = \{\text{tossing 2 coins}\} = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$

What's the $\sigma$-field of $\Omega \times \Omega$? Is it $\mathcal{F} \times \mathcal{F}$?

$\mathcal{F} \times \mathcal{F} = \{\emptyset, \Omega, \{H\}, \{T\}\} \times \{\emptyset, \Omega, \{H\}, \{T\}\}$

\[\emptyset \times \{H\} \quad \emptyset \times \{T\} \quad \Omega \times \emptyset \quad \Omega \times \{H\} \quad \Omega \times \{T\}\]

$= \{\emptyset, \Omega \times \{H\}, \{H, \Omega\}, \{H, H\}, \{H, T\}, \{T, H\}, \{T, T\}\}$

Using $P(J) = P(\Omega \times \{T\}) = P(\Omega) \cdot P(\{T\}) = 1 \cdot \frac{1}{2} = \frac{1}{2}$

$U = \{(H, H), (T, T)\} = \{\text{both tosses have the same}\}$

$\therefore \mathcal{F} \times \mathcal{F}$ is not a $\sigma$-field!

*This example is still easy because the $\sigma$-field should be $2^{\Omega \times \Omega}$ which is finite.

But what happen when $\Omega = [0, 1]$ Borel measure?

Soln: $\mathcal{F} \times \mathcal{F}$ is a semi-algebra.

Apply CET on $\mathcal{F} \times \mathcal{F}$ and $P: \mathcal{F} \times \mathcal{F} \to [0, 1]$ based independence.

$P(J \times J_2) = P(J) \cdot P(J_2)$

$\Rightarrow (\Omega \times \Omega, \mathcal{B}(\mathcal{F} \times \mathcal{F}), P^*)$ product measure \(\text{DONE!}\)
Basic Properties of Probability

Given \((\Omega, \mathcal{F}, P)\)

1. \(\emptyset \in \mathcal{F}\)
2. If \(A_1, A_2, \ldots \in \mathcal{F}\) \(\Rightarrow \bigcap_{i=1} \in \mathcal{F}\)
3. \(P(\emptyset) = 0\)
4. If \(A \subset B\), \(A, B \in \mathcal{F}\) \(\Rightarrow P(A) \leq P(B)\)

5. Inclusion-Exclusion Principle

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)
\]

6. Independence \(A, B \in \mathcal{F}\) are independent \((A \perp B)\)

\[
\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)
\]

\[
P(A \cap B) = 0 \quad \text{WRONG}
\]

\[\begin{array}{c}
\text{A} \\
\hline \\
\text{B} \\
\hline \\
\Omega
\end{array}\]

\[
P(A) = \frac{1}{2} \quad \text{A = top half}
\]

\[
P(B) = \frac{1}{2} \quad \text{B = right half}
\]

\[A \perp B \text{ because } P(A \cap B) = \frac{1}{4} = P(A) \cdot P(B)\]
Properties

0. \(A_1, A_2, \ldots, A_N\) \rightarrow total\ independence

\(\text{Any subset } J \subset \{1, 2, \ldots, N\}\)

\[ P(\bigcap_{j \in J} A_j) = \prod_{j \in J} P(A_j) \]

\[ P(A_i \cap A_j) = P(A_i)P(A_j) \quad \text{for } i \neq j \]

\(\text{Not the same (HW)}\)

0. For any \(A \in J\), \(A \subseteq \Omega\) and \(A \not\subseteq \phi\)

0. Conditional probability

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

Ex: Lunch date with Pat at 12:30 pm and Pat didn’t show up

\(A = \{\text{Pat is in a horrible accident}\}\) \(P(A) = 0.001\)

\(B = \{\text{your watch is broken}\}\) \(P(A \mid B) = 0.0000001\)

a) Chain of conditional probability

Let \(D = A \cap B\)

\[ P(A \cap B \cap C) = P(C \mid A \cap B)P(A \cap B) \quad \therefore \quad P(C \mid D) = \frac{P(C \cap D)}{P(D)} \]

\[ = P(C \mid A \cap B) \cdot P(B \mid A) \cdot P(A) \]

\[ P(\bigcap_{i=1}^{N} A_i) = P(A_N \mid \bigcap_{i=1}^{N-1} A_i) \cdot P(A_{N-1} \mid \bigcap_{i=1}^{N-2} A_i) \cdots \cdot P(A_1) \]
b) Bayes Rule

\[
P(A|B) = \frac{P(AB)}{P(B)}
= \frac{P(B|A) \cdot P(A)}{P(A) \cdot P(B)}
= \frac{P(B|A) \cdot P(A)}{P(B) + P(A^c|B) \cdot P(A^c)}
\]

Bayes Rule

ex

\[
\begin{align*}
    A &= \{\text{Man-made global warming}\} \\
    B &= \{\text{Ozone depletion, north pole rising}\}
\end{align*}
\]

\[
P(A|B) \text{ important question to ask but very difficult to prove}
\]

\[
P(B|A) \text{ is "easy" by computer simulation.}
\]