1. A continuous-time signal $x(t)$ is sampled with sampling period $T=0.5$ second.
   
   a. What is the period (in radian/second) of its Fourier Transform?

   The period is just the sampling frequency $\omega_s = \frac{2\pi}{T} = 4\pi = 12.57$ rad/s

   b. If the highest frequency component of $x(t)$ is 5 Hz, can we fully recover the original continuous-time signal based on the discrete-time sample $x(nT)$? Justify your answer.

   Since the sampling frequency is $1/0.5$ or 2 Hz, the highest frequency that can pass through without aliasing is $2/2 = 1$ Hz according to the Nyquist Theorem. Thus we cannot recover the original signal.

   c. Give three different types of D/A methods and describe their relative performance in terms of complexity and reconstruction fidelity.

   In decreasing order of complexity and reconstruction accuracy:
   
   Ideal LowPass > RC filter > Linear Interpolation > Sample-and-Hold

   d. Given the discrete-time input $x(nT)$ to a LINEAR INTERPOLATION D/A converter, sketch the output reconstructed continuous signal $y(t)$ on top of $x(nT)$.

2. Sampling and Interpolation

   a. (4 points) State the Nyquist Theorem

   Nyquist Theorem states that a continuous-time band-limited signal can be perfectly reconstructed from its discrete samples if the sampling frequency is twice the bandwidth.

   b. (7 points) Explain why linear interpolation is a better D/A technique than sample-and-hold.

   Linear interpolation is better than sample-and-hold as the frequency response of linear interpolation decays faster than sample-and-hold and thus better suppress the high frequency components introduced by sampling.

   c. (7 points) A continuous-time signal $x_c(t) = \cos(4000\pi t)$ is sampled with sampling period $T$ to obtain the discrete-time signal $x_d(nT) = x_d(n) = \cos(\pi n/3)$. Determine a choice of $T$ that is consistent with this information.
Since \( x_d(n) = x_c(nT) \), we have \( 4000\pi n T = \pi n / 3 \Rightarrow T = \frac{1}{12000} \)

d. (7 points) Continued from part c., is your choice for \( T \) unique? If so, explain why. If not, specify another choice of \( T \).

\( T \) is not unique because \( \cos(4000\pi n T) = \cos(\frac{\pi n}{3} + 2m\pi) \) for any integer \( m \). In particular we can choose \( m = n \), and have \( 4000\pi n T = \pi n / 3 + 2n\pi \Rightarrow T = \frac{7}{12000} \)

3. In the following system, two continuous-time functions \( x_1(t) \) and \( x_2(t) \) are MULTIPLIED and the product \( w(t) \) is sampled with sampling period \( T \).

\[
\begin{align*}
&x_1(t) \quad \downarrow \quad \diamond \quad \downarrow \quad w(t) \\
x_2(t) &\quad \downarrow \quad \diamond \quad \downarrow \quad w_s(t)
\end{align*}
\]

If the spectrums of \( x_1(t) \) and \( x_2(t) \) are given as follows, determine the maximum sampling period \( T \) such that \( w(t) \) is recoverable from \( w_s(t) \) through the use of an ideal lowpass filter.

\[
\begin{align*}
&X_1(f) \quad \text{[-10, 10 Hz]} \\
&X_2(f) \quad \text{[-5, 5 Hz]}
\end{align*}
\]

Since multiplication in time domain is equivalent to convolution in frequency domain, the signal \( w(t) = x_1(t)x_2(t) \) is bandlimited within \([-10-5 \text{ Hz}, 10+5 \text{ hz}] \) or \([-15 \text{ Hz}, 15\text{Hz}] \). Thus the maximum sampling period \( T = 1/(15 \cdot 2) = 1/30 \) seconds.

4. Discrete-Time Fourier Transform

a) Given the DTFT of a LTI system is \( H_d(\omega) = \sum_{n=0}^{\infty} h(nT)e^{-j\omega Tn} \), show that if the impulse response \( h(nT) \) is real, we have \( H_{d}(-\omega) = \overline{H_{d}(\omega)} \), i.e. the negative frequency content can be deduced by taking the conjugate of the positive frequency content.

We can go straight from the definition:

\[
\begin{align*}
H(-\omega) &= \sum_{n=0}^{\infty} h(nT)e^{j\omega Tn} \\
&= \overline{\sum_{n=0}^{\infty} h(nT)e^{-j\omega Tn}} = \overline{H(\omega)}
\end{align*}
\]
Sample Midterm

b) If the input to $H(\omega)$ is $x(nT) = \cos(\omega_0 n T) = \frac{1}{2}(e^{j\omega_0 n T} + e^{-j\omega_0 n T})$, use part a) to show that the output is given by

$$y(nT) = |H(e^{j\omega_0 T})| \cos(\omega_0 n T + \angle H(e^{j\omega_0 T}))$$

As $x(nT) = \cos(\omega_0 n T) = \frac{1}{2}(e^{j\omega_0 n T} + e^{-j\omega_0 n T})$ and we know that the complex exponential is the “eigen-signal” of any LTI system. Thus, the output

$$y(nT) = \frac{1}{2} \left( |H(\omega_0)| e^{j\omega_0 n T} + H(\omega_0)e^{-j\omega_0 n T} \right)$$

As $y(nT) = \cos(\omega_0 n T) + \frac{1}{2}(\exp(j(\omega_0 n T + \angle H(\omega_0)) + \exp(j(-\omega_0 n T - \angle H(\omega_0))))$

$$= \frac{1}{2} \left( |H(\omega_0)| \cos(\omega_0 n T + \angle H(\omega_0)) \right)$$

5. Laplace Transform

a. No need to compute the actual coefficients, write down the INVERSE LAPLACE TRANSFORM of the followings:

i) $X(s) = \frac{7s^3 + 20s^2 + 33s + 82}{(s^2 + 4)(s + 2)(s + 3)}$

$$X(s) = \frac{A}{s - j2} + \frac{\bar{A}}{s + j2} + \frac{B}{s + 2} + \frac{C}{s + 3}$$

$$x(t) = Ae^{j2t} + \bar{A}e^{-j2t} + Be^{-2t} + Ce^{-3t}$$

ii) $X(s) = \frac{s^2(s + 9)}{(s + 3)^3(s + 1)}$

$$X(s) = \frac{A}{(s + 3)^3} + \frac{B}{(s + 3)^2} + \frac{C}{s + 3} + \frac{D}{s + 1}$$

$$x(t) = \frac{A}{2}t^2 e^{-3t} + Bte^{-3t} + Ce^{-3t} + De^{-t}$$

b. A dynamic system is governed by the following differential equation with initial conditions $\frac{dy}{dt}\bigg|_{t=0} = 1, y(0) = 0$

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} y(t) + y(t) = x(t)$$

Given the input $X(s) = 1$. Write down the Laplace transform of the zero-state response and the zero-input response.

Writing the differential equation in complex domain, we have:
Sample Midterm

\[ s^2 Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = X(s) \]
\[ (s^2 + 2s + 1)Y(s) = sy(0) + y'(0) + 2y(0) + X(s) \]

\[ Y(s) = \frac{sy(0) + y'(0) + 2y(0)}{s^2 + 2s + 1} + \frac{X(s)}{s^2 + 2s + 1} \]

The first term depends only on the initial condition so it is the zero-input response. Putting in the numerical value we have:

\[ Y_{ZIR}(s) = \frac{1}{s^2 + 2s + 1} \]

The second term depends only on the input so it is the zero-state response. Putting in the numerical value we have:

\[ Y_{ZSR}(s) = \frac{1}{s^2 + 2s + 1} \]

6. Discrete Fourier Transform

Let \( x(n) \) and \( y(n) \) be two three-point sequence:

\[
\begin{align*}
  x(n) &= \begin{cases} 
    1 & \text{for } n = 0 \\
    2 & \text{for } n = 1 \\
    1 & \text{for } n = 2 
  \end{cases} \\
  y(n) &= \begin{cases} 
    -1 & \text{for } n = 0 \\
    2 & \text{for } n = 1 \\
    1 & \text{for } n = 2 
  \end{cases}
\end{align*}
\]

Compute the 5-point DFT \( X(k) \) for \( x(n) \). You do not need to simplify your answers.

\[ X(0) = 1 + 2 + 1 = 4 \]
\[ X(1) = 1 + 2 \cdot \exp\left(-j \frac{2\pi}{5}\right) + \exp\left(-j \frac{4\pi}{5}\right) \]
\[ X(2) = 1 + 2 \cdot \exp\left(-j \frac{4\pi}{5}\right) + \exp\left(-j \frac{8\pi}{5}\right) \]
\[ X(3) = 1 + 2 \cdot \exp\left(-j \frac{6\pi}{5}\right) + \exp\left(-j \frac{12\pi}{5}\right) \]
\[ X(4) = 1 + 2 \cdot \exp\left(-j \frac{8\pi}{5}\right) + \exp\left(-j \frac{16\pi}{5}\right) \]

7. Assume \( x(t) = a_1(t)x_1(t) + a_2(t)x_2(t) \), \( x_1(s) = L[x_1(t)] \) and \( x_2(s) = L[x_2(t)] \).

a. Is \( X(s) = L[x(t)] \) equal to \( a_1(t)X_1(s) + a_2(t)X_2(s) \) and why?

No. The Laplace transform is a function of \( s \) not \( t \), and thus cannot contain terms like \( a_1(t) \) and \( a_2(t) \). – 2 points

b. If \( A_1(s) = L[a_1(t)] \) and \( A_2(s) = L[a_2(t)] \), is \( L[x(t)] = A_1(s)X_1(s) + A_2(s)X_2(s) \) and why?


Sample Midterm

No. The linearity property can only be applied to constant coefficients, not time-dependent ones. – 2 points

c. If \( x(t) = e^{-2t}u(t-2) \), which of the following is the Laplace transform of \( x(t) \)?

(a) \( \frac{1}{s-2} e^{-s} \), (b) \( \frac{1}{s+1} e^{-2s} \), (c) \( \frac{1}{s-1} e^{-s} \), (d) none of the above.

We can break it down into two steps:

\( e^{t-2} u(t-2) \) \( \Leftrightarrow \) time delay \( \Leftrightarrow \) \( e^t u(t) \) \( \Leftrightarrow \) multiplication of exponential \( \Leftrightarrow \) \( u(t) \)

Since \( L[u(t)] = \frac{1}{s} \), the complex shifting theorem tells us that \( L[e^t u(t)] = \frac{1}{s-1} \).

Finally, time delay corresponds to multiplication of exponential in s-domain, i.e. \( L[e^{t-2}u(t-2)] = \frac{e^{-2s}}{s-1} \) or (c).

8. Laplace Transform

a. \( x(t) \)'s Laplace transform is \( X(s) = \frac{s+4}{s^2+3s+2} \). Draw its ROC.

After simplification, \( X(s) = \frac{1}{(s+1)(s+2)} \). The poles of \( X(s) \) are -1 and -2. (2 pts)

The ROC looks like

b. Find the differential equation relating the input \( x(t) \) and the output \( y(t) \) if the transfer function of the system is given as follows:

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 2s + 1} \]

\[ H(s) = \frac{1}{s^2 + 2s + 1} \Rightarrow s^2 Y(s) + 2sY(s) + Y(s) = X(s) \]

By the definition of transfer function, all initial conditions are assumed to be zero.

Taking inverse Laplace transform, we get

\[ \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + y(t) = x(t) \]